(1) (10%) Show that any equation $y''+2p(x)y'+q(x)y=0$ can be reduced to the "canonical form" $v''+r(x)v=0$ by a change of dependent variable $y=a(x)v$; that is, determine $a(x)$ and $r(x)$.

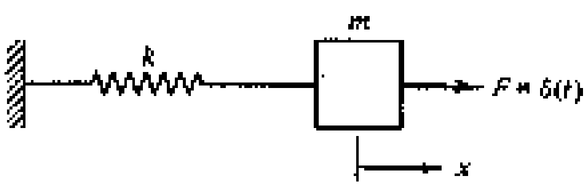
(2) (10%) If the complementary solution of $y''+p(x)y'+q(x)y=f(x)$ is $Ay_1(x)+By_2(x)$, **DERIVE** the general solution (Note that your solution may include the Wronskian).

(3) (10%) If $\hat{A}$ is a constant vector and $r=\sqrt{x^2+y^2+z^2}$, show that

$$\hat{A}\bullet\nabla\left(\frac{1}{r}\right)=-\frac{\hat{A}\bullet\hat{r}}{r^3} \quad \text{and} \quad \nabla(\hat{A}\bullet\hat{r})=\hat{A}$$

(4) (7%) What (acute) angle does $\hat{A}=\hat{i}-2\hat{k}$ make with the normal to the plane containing the vectors $\hat{B}=\hat{j}-\hat{k}$ and $\hat{C}=\hat{i}+\hat{j}+\hat{k}$?

(5) (13%) As shown in the figure, the mass m is initially at rest. At $t=0$ a unit impulsive force is applied. Thus $m\ddot{x}+kx=\delta(t)$, subject to the initial conditions $x(0)=\dot{x}(0)=0$. Solve for $x(t)$ by Laplace transform. Is $\dot{x}(t)$ continuous at $t=0$? Comment and resolve.



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6. Find all values of the followings:

(a) (5%) i^i

(b) (5%) $\log\left(i^z\right)$

where $i = \sqrt{-1}$ and $z = x + iy$.

7. (10%) Let

$$A = \begin{bmatrix} 3 & 4 \\ -5 & -5 \end{bmatrix}$$

Find all eigenvalues and the corresponding eigenvectors of A viewed as matrix over (a) the field of real number, (b) the field of complex number.

8. Consider the eigenvalue problem in $0 < x < 1$

$$y''(x) + \lambda\ y(x) = 0,$$

with mixed boundary conditions, $y(0) = 0$, $y'(0) = y(1)$.

(a) (10%) Show that there is only one real eigenvalue. What is the corresponding eigenfunction?

(b) (5%) Why does the Sturm-Liouville theorem fail which states that there are infinitely many real eigenvalues?

9. (15%) Solve the one-dimensional heat equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

with the initial point source, $T(x,0) = Q\delta(x)$, and the boundary conditions, $T(x \to \pm\infty, t) = 0$, where α and Q are constants.