1. (15%) Given two solutions of a nonhomogeneous second order linear ordinary differential equation, can you find a particular solution of the corresponding homogeneous equation? The general solution of the homogeneous equation? Show your work step by step.

2. Briefly describe the followings:

   (a) (4%) The dimension and a basis of a linear vector space.

   (b) (3%) Consider a continuously differentiable scalar function $f(x, y, z)$. What's the direction and the magnitude of the maximum increase of f at some point (x_0, y_0, z_0)?

   (c) (5%) The necessary and sufficient conditions for $\oint_C \vec{v} \cdot d\vec{r} = 0$, where $\vec{v}$ is a continuous vector field, C is any close curve in a domain D (simply-connected or multiply-connected), and $\vec{r}$ is the position vector of the points on C. Explain (without proof) the reason briefly.

   (d) (8%) Consider a set of functions $\{\phi_1(x), \phi_2(x), \cdots, \phi_n(x), \cdots\}$ orthogonal with respect to a weighting function $p(x)$ on $[a, b]$. What do we mean that it is a complete set (or a total set)? What is the importance of the completeness?

3. (15%) Find the eigenvalues and the corresponding eigenfunctions of the following boundary value problem. Work out the eigenfunction expansion of the given function, $f(x)$. Determine what the expansion converges to on the relevant interval.

$$\begin{cases} y'' + 2y' + (1 + \lambda)y = 0, \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

$$f(x) = \begin{cases} 0, & 0 \le x \le 1/2 \\ 1, & 1/2 < x \le 1 \end{cases}$$

(背面仍有題目,請繼續作答)

4. Please evaluate the following integral. Please remember to draw your integral contour and show everything in detail. (15%)

$$I=\int_{-\infty}^{\infty}\frac{\cos x}{\pi^2-4x^2}dx$$

5. Find the Fourier transform of e^{-ax^2}, where $a>0$. (10%)

6. Please write a series solution of the following boundary value problem.(15%)

$$\frac{\partial^2 y}{\partial t^2}=25\frac{\partial^2 y}{\partial x^2} \qquad (0<x<\pi,\ t>0),$$
$$y(0,t)=y(\pi,t)=0 \qquad (t>0),$$
$$y(x,0)=\sin(2x) \qquad (0<x<\pi),$$
$$\frac{\partial y}{\partial t}(x,0)=\pi-x \qquad (0<x<\pi)$$

7. Find the spectrum and eigenvectors of the following matrix. (Please show the details of your work.) (10%)

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 \end{bmatrix}$$