Problem 1 (5%)

Two different chemical solutions are pumped into a container of volume 100 ℓ, each at the rate of 5 ℓ/sec, and thoroughly mixed solution is pumped out of the container at the rate of 10 ℓ/sec. The inflow concentration of Chemical 1 is q_1 (kg/ℓ) and the inflow concentration of Chemical 2 is q_2 (kg/ℓ). Denote the mass of Chemical 1 (in the container) by x_1 (kg) and the mass of Chemical 2 by x_2 (kg). A catalyst in the container transforms Chemical 1 into Chemical 2 at the rate of $0.4x$ (kg/sec).

(a) Formulate the 2×2 linear system of first order ODEs that x_1 and x_2 satisfy.

(b) Without finding the general solution, determine the steady-state solution for x_1 and x_2.

Problem 2 (10%)

A solution to the unforced equation $\ddot{x} + b\dot{x} + 4x = 0$ is observed to take the value $x = 0$ for $t = 0$ and next at $t = \pi$. (Assume that $b \geq 0$.)

(a) Is the equation underdamped, critically damped, or overdamped? Explain.

(b) What is the value of the damping constant b?

Problem 3 (10%)

A certain matrix $\mathbf{A}$ has eigenvalues 1 and -1, with eigenvectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{respectively.}$$

(a) Find the solution for the initial value problem

$$\dot{\mathbf{u}} = \mathbf{A}\mathbf{u} \quad \text{with} \quad \mathbf{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(b) Calculate $\mathbf{A}^{9999}$.

Problem 4 (10%)

The temperature T at a point (x, y, z) in space is inversely proportional to the square of the distance from (x, y, z) to the origin. It is known that $T(0, 0, 1) = 500$.

(a) Find the rate of change of T at $(2, 3, 3)$ in the direction of $(3, 1, 1)$.

(b) In which direction from $(2, 3, 3)$ does the temperature T increase most rapidly?

(c) At $(2, 3, 3)$ what is the maximum rate of change of T?

(背面仍有題目,請繼續作答)

5. Find Laplace Transform of $y(t)$ that satisfies the equation as follows:

$$d^2y/dt^2 + y = e^{-t}\int_0^t t\sin 2t\ dt$$ **(12 %)**

6. Find the solution of differential equation in power series, $x \neq 0$.

$$xd^2y/dx^2 + dy/dx - y = 0$$ **(15 %)**

7. Find the Fourier Series of $f(x)$ that is defined as follows: **(8 %)**

$$f(x) = |x| \quad \text{where} \quad -1 \le x \le 1$$

8. The function f(z) = u(r, θ) + iv(r, θ) is given. Derive the Cauchy-Riemann equations in polar coordinates as **(15%)**

$$\frac{\partial u}{\partial r} = \frac{\partial v}{r\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{\partial u}{r\partial \theta}$$

9. Solve the following problem (15%)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{for} \quad 0 < x < L, \quad t > 0$$

with

$$\frac{\partial u}{\partial x}(0,t) = -1, \quad \frac{\partial u}{\partial x}(L,t) = 0$$

and

$$u(x, 0) = 0$$