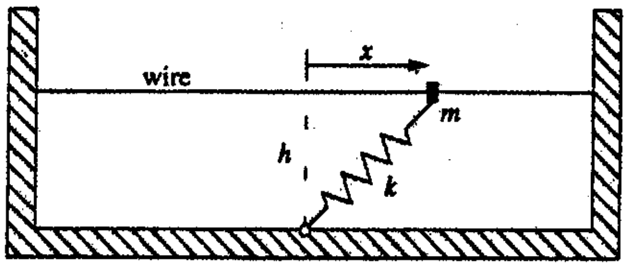
本試題是否可以使用計算機： ☑可使用 ， □不可使用 （請命題老師勾選）

1.



A bead of mass m is constrained to slide along a straight horizontal wire. A spring of relaxed length L_0 and spring constant k is attached to the mass and to a support point a distance h from the wire (as sketched above). Suppose also that the motion of the bead is opposed by a viscous damping force $b\dot{x}$.

(a) Derive the *nonlinear* differential equation for the motion of the bead, and find *all* possible *equilibrium points* as functions of k, h, m, b, and L_0. (5%)

(b) The equation of motion derived in (a) can be linearized about $x = 0$ (which is obviously an equilibrium point) to yield

$$m\ddot{x} + b\dot{x} + k(1 - L_0/h)x = 0.$$

Suppose that $m = 0$. Integrate the above linearized equation of motion with an appropriate initial condition, and hence discuss the *stability* of the obvious equilibrium point $x = 0$. (5%)

(c) Now, suppose that $b = 0$ (but $m \neq 0$). Use *Laplace transform* to solve the linearized equation of motion in (b) with the initial conditions $x(0) = \epsilon$ and $\dot{x}(0) = 0$. (10%)

2. (a) Complete the matrix $\mathbf{A}$ (fill in the two blank entries) so that $\mathbf{A}$ has eigenvectors $\mathbf{x}_1 = (3,1)^T$ and $\mathbf{x}_2 = (2,1)^T$:

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ & \end{bmatrix} \quad (5\%).$$

(b) Find a matrix $\mathbf{B}$ with those same eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$, and with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$. Also calculate $\mathbf{B}^{10}$. (10%)

3. Given a two-dimensional vector field

$$\mathbf{V} = \frac{x^2 - y^2}{(x^2+y^2)^2}\mathbf{i} + \frac{2xy}{(x^2+y^2)^2}\mathbf{j}.$$

(a) Calculate by *direct integration* the *circulation* of $\mathbf{V}$ around the circular closed contour of radius 1 centered at the origin. To earn full credits, detail your calculations. (6%)

(b) Calculate the same circulation as in (a) by use of *Stokes's theorem*. Again, to earn full credits, detail your calculations. (9%)

（背面仍有題目,請繼續作答）

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4. Show that substitution of $u = F(r,\theta)\, G(t)$ into the wave equation:

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right)$$

leads to

$$\ddot{G} + \lambda^2 G = 0 \qquad \text{where } \lambda = ck$$

and

$$F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\theta\theta} + k^2 F = 0$$

(25%)

5. Prove the following series converges uniformly in the given region.

$$\sum_{n=0}^{\infty} \frac{z^n}{|z|^{2n} + 1}, \qquad 2 \le |z| \le 4$$

where z is the complex variable.

(25%)