1. (30%) Figure 1 shows an inverted pendulum pinned on a trolley (with mass m_t) that moves on a level, frictionless table. The homogenous pendulum has mass m, length $2L$, and centroidal moment of inertia I.
   (a) [15%] Derive the equation of motion of the pendulum using Newton's laws of motion. Express it in terms of symbols provided in Figure 1 only.
   (b) [15%] The approach you apply to derive the equation in (a) is originated from classical mechanics. An alternative approach using the conservation of energy and momentum is also possible. Different from classical mechanics that deals with vector equations formulated with aid of free body diagrams. This alternative approach deals with scalar equations that come from the kinetic and potential energy of a mechanical system. A simplified yet detailed instruction to apply this approach is as follows. Please re-derive the equation of motion of the pendulum using this step-by-step instruction.
      (1) Obtain the kinetic energy T and potential energy V of the pendulum. The gravitational force is pointing downward with gravitational constant g.
      (2) Differentiate the potential energy with respect to θ.
      (3) Differentiate the negative of kinetic energy (-T) with respect to θ.
      (4) Differentiate the kinetic energy with respect to $\dot{\theta}$. (Treat $\dot{\theta}$ and θ as two independent variables, for example, $d(\cos\theta)/d\dot{\theta}=0$ and $d\dot{\theta}/d\theta=0$). After that, differentiate it again with respect to time t.
      (5) Set the summation of the results in (2)~(4) equal to zero. You then get the equation governing the motion of the pendulum.

2. (20%) A homogenous cube of side a and mass M slides on a level, frictionless table with velocity $\mathbf{v}_0$. See Figure 2. It strikes a small lip on the table at A of negligible height. Find the velocity of the center of mass just after impact if the coefficient of restitution is unity. (The centroidal moment of inertia of a cube is $Ma^2/6$)

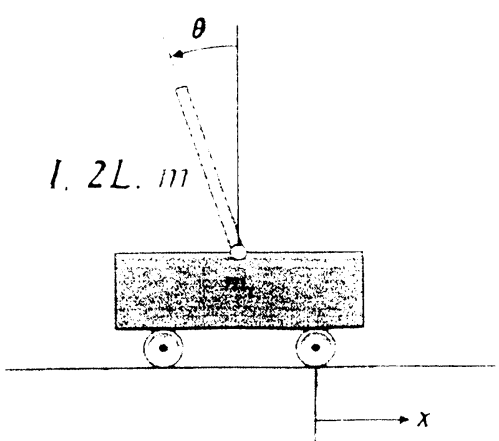


Figure 1

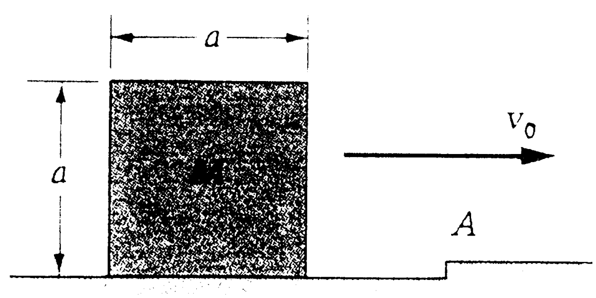


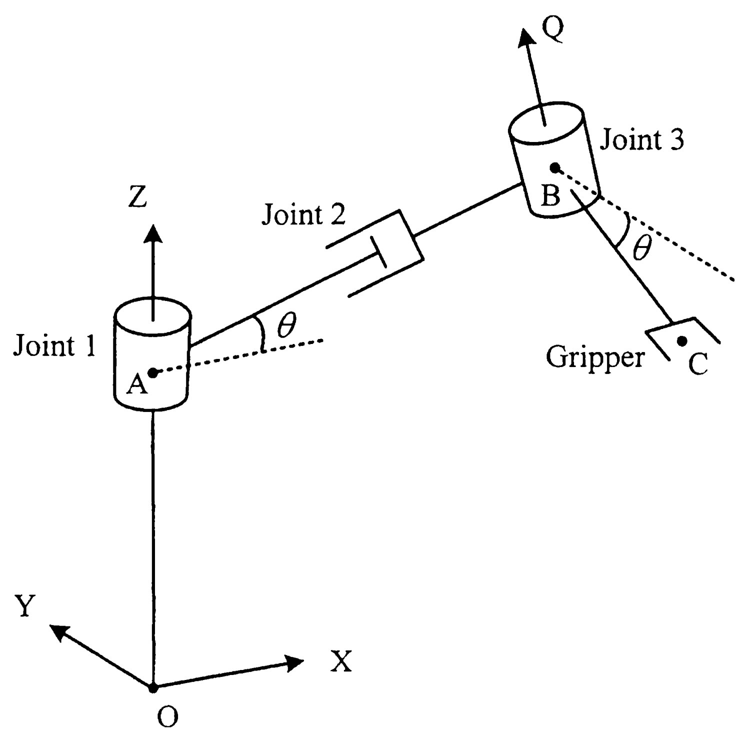
Figure 2

（背面仍有題目,請繼續作答）

3. (30%) Consider a three-degree-of-freedom robot used to transport steel balls in a factory. A schematic drawing of the robot is shown in the figure. The robot comprises a revolute joint (Joint 1), a prismatic joint (Joint 2), another revolute joint (Joint 3), and a gripper, which is used to grasp a steel ball. At the instant shown in the figure, the angular velocity at Joint 1 is 1 rad/sec (constant, no angular acceleration), measured about the positive direction of the Z axis. The (relative) velocity measured at Joint 2 is 3 m/sec, and the (relative) acceleration at Joint 2 is 4 m/sec^2. The (relative) angular velocity measured at Joint 3 is 2 rad/sec (constant), about the positive direction of axis Q. The axis of Joint 3 (axis Q) is on the X-Z plane and perpendicular to line AB. Line AB is on X-Z plane too, and the angle that line AB makes with the X axis is $\theta = 30$ degrees. Line BC is parallel to the Y-Z plane, and the angle that line BC makes with the Y axis is also $\theta = 30$ degrees. The distance between O and A is 2 m, the distance between A and B is 2 m too, and that between B and C is 1 m. The X-Y plane is the floor.

(a) You are asked to use vector operations to determine the velocity and acceleration of point C. Do not use graphical methods.

(b) A 10 kg steel ball, whose center is coincident with C, slips out of the gripper under the condition calculated in part (a). The motion of the ball is assumed to be the same as that of the gripper when the ball slips away from the gripper. Where will the steel ball first hit the floor? Neglect the radius of the steel ball in your calculation.



4. (20%) It is well known that the velocity vector of a point on a rigid body is obtained from the differentiation of the position vector of the point of interest. However, it is a misconception that the angular velocity vector of a rigid body can be derived similarly, by differentiating a vector. Rather, the information of angular velocity is obtained from the differentiation of a rotation matrix. Therefore, it is natural to ask why we use angular velocity vectors instead of matrices. In what follows, you will be guided through the derivation, and you have to follow the instructions and record equations on your answer sheet.

There are many ways to obtain a rotation matrix. Euler angles, for example, are frequently used to describe and derive a rotation matrix. A rotation matrix is an orthonormal three-by-three matrix that transforms one position vector to another, and one of the most important properties of an orthonormal matrix is that its inverse matrix is identical to its transpose matrix. Take a note of this property in your answer sheet now by using the letter A to denote the orthonormal matrix. This equation is numbered as Equation (1). Let us move on with the discussion. The angular velocity of a rigid body is indeed a matrix, called the angular velocity matrix, and it is the product of the time-derivative of the rotation matrix A and the transpose of the rotation matrix. Please record this statement on your answer sheet as Equation (2). Let the angular velocity matrix be denoted by B.

It can be shown that the angular velocity matrix is a skew symmetric matrix. As you may have recalled, the transpose of a skew symmetric matrix is equal to the negative of the skew symmetric matrix. As a result, the diagonal elements of a skew symmetric matrix all vanish. Now, you can write down the angular velocity matrix as a skew symmetric matrix. This is numbered as Equation (3). To be more specific, you have to explicitly write down all the 9 elements of the three-by-three skew symmetric matrix. Let the elements of B be denoted by b_{ij}, where i and j denote the row and column numbers, respectively. Note that there are only three independent elements among all of the 9 elements, considering the properties of a skew symmetric matrix.

Now we are ready to relate the angular velocity matrix to the commonly used angular velocity vector. First, expand the product of the skew symmetric matrix B and a position vector, denoted by a column vector r whose components are r_1, r_2, and r_3. The result should be numbered as Equation (4). Next, form the angular velocity vector, which is represented by a column vector ϖ, as follows. The first component of the angular velocity vector is the negative of the element at row 2 and column 3 of B, the second component is the element at row 1 and column 3 of B, and the third component is the negative of the element at row 1 and column 2 of B. Write down the expression of the angular velocity vector and number it as Equation (5). Finally, expand the cross product of the angular velocity vector and the position vector r and number it as Equation (6).

You can now compare the result in Equation (6) to the product of the angular velocity matrix and the position vector, as recorded in Equation (4). If you follow the instructions and derive the equations correctly, you should have identical results. In conclusion, although the angular velocity vector lacks geometrical meanings, it serves well in calculations for the purpose of avoiding matrix operations.