(1) Solve ordinary differential equation

$$y''+4y'+4y = f(t), \quad t \geq 0$$

with initial conditions $y(0) = y'(0) = 0$. Also, $f(t)$ is defined as

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}, \quad (15\%)$$

(2) Let a matrix

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}.$$

Find (i) eigenvalues of the matrix. (5%)

(ii) associated eigenvectors of (i). (10%)

(3) Consider the second order ordinary differential equation

$$x y'' + y' + \lambda^2 x y = 0, \quad 1 \leq x \leq 2$$

where λ is the undetermined parameter. The boundary conditions are $y(1) = y(2) = 0$.

(i) Show above equation may be transformed to

$$s \frac{d^2 y}{ds^2} + \frac{dy}{ds} + s y = 0, \text{ with } s = \lambda x, \ (5\%)$$

(ii). Find the equation which can be used to find the value of λ to make the solution not trivial (i.e. $y \neq 0$). (10%)

(4) A particle moves with the locus given by

$$x = t, \quad y = 2t, \quad z = 4t, \quad 0 \leq t \leq 2.$$

A force is applied on the particle with

$\vec{F} = x^2 \vec{i} - 2xyz \vec{j} + z^2 \vec{k}$, $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors along x, y, z axises, respectively.

(i) Find the instantaneous velocity (5%)

(ii). Find the work done on the particle during $0 \leq t \leq 2$. (10%)

(5). The unsteady state conduction in an infinite large plate with thickness to be 1 can be described by

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

with boundary condition

$$x=0, \quad T(t,0)=1$$
$$x=1, \quad T(t,1)=1$$

and initial condition

$$t=0, \quad T(0,x)=2x.$$

Use separation of variables method to solve the problem. (20%)

(6). For the function defined in the complex plane

$$f(z)=\frac{2z+1}{z^3-iz^2+6z}, \quad i=\sqrt{-1}, \quad z \text{ is the complex variable.}$$

(i) Show $f(z)$ is analytic in the domain surrounded by a circle of radius 2 about the point $(1,-2)$ in the complex plane. (10%)

(ii). Find the integral $\int_C f(z)\,dz$, along the closed path C in the countercurrent sense, where C is the circle of radius 2 about the point $(0,3)$ in the complex plane. (10%)