1. Solve the following ordinary differential equations:

(a) $xy'' + y' = y'^2$ (5%)

(b) $\begin{cases} x' = -2x + y \\ y' = -4x + 3y + 10\cos t \end{cases}$ (5%)

(c) $y'' - 2y' + y = e^x + x$ (5%)

(d) $(x^2D^2 - 2xD + 2)y = x^3\cos x$ (5%)

2. Answer the following questions: (20%)

(1) Wronskian $= W(y_1,y_2,\ldots,y_n) = ?$ for linear dependence and independence of functions $y_1,y_2,\ldots,y_n$.

(2) What's the radius of convergence of the series $\sum_{m=0}^{\infty} x^m/m!$?

(3) Legendre polynomial of degree n, $P_n(1) = ?$

(4) Gamma function, $\Gamma(\alpha+1) = ?$ for $\alpha>0$.

(5) An orthonormal set $g_1,g_2,\ldots$ on an interval $a\le x\le b$, $(g_m,g_n) = ?$ $m=1,2,\ldots;n=1,2,\ldots$

(6) $L^{-1}[\ 1\ ] = ?$

(7) Does $1*f = f$ in general?

(8) Does $\bar{A}\bar{B} = \bar{0}$ imply $\bar{A} = \bar{0}$ or $\bar{B} = \bar{0}$?

(9) $\bar{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\bar{A}^{-1} = ?$

(10) Jacobian $= J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = ?$ where x,y, are rectangular coordinates and r,θ, are polar coordinates.

3. Granted sufficient differentiability, find

(a) $\operatorname{div}(\operatorname{curl} \vec{v})$ (5%)

(b) $\operatorname{curl}(\operatorname{grad} f)$ (5%)

4. Solve the initial value problem

$$y'' + 3y' + 2y = 1 - u(t - 1),\ y(0)=0,\ y'(0)=1, \qquad (10\%)$$

where u is the unit step function.

5. Using the Fourier integral representation, show that

$$\int_0^{\infty} \frac{w\sin wx}{k^2+w^2}\,dw = \frac{\pi}{2} e^{-kx} \quad , \quad x>0,\ k>0 \qquad (10\%)$$

6. In a body heat will flow in the direction of decreasing temperature. It can be shown that the velocity $\vec{v}$ of the heat flow in a body is of the form

$$\vec{v} = -K \operatorname{grad} U$$

where U(x,y,z,t) is temperature, t is time and K is the thermal conductivity of the body. Using this information, set up the mathematical model of heat flow, the so-called heat equation, by means of divergence theorem of Gauss.

$$\frac{\partial U}{\partial t} = c^2 \nabla^2 U , \quad c^2 = \frac{K}{\sigma \rho} , \quad \sigma: \text{specific heat},\ \rho: \text{density} \qquad (10\%)$$

7. (a) Find the temperature U(x,t) in a bar of length L that is perfectly insulated, also at the ends at x=0 and x=L, assuming that U(x,0)=f(x), by the method of separating variables. (15%)

(b) What's the corresponding eigenvalue problem? (5%)