(乙)

1. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + \tan^{-1}x$, where $\tan^{-1}: \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is defined by $\tan^{-1}x = y$ if and only if $\tan y = x$. Graph $y = f(x)$ by giving its essential shape and pointing out its asymptotes and points of inflection. 15%

2. Find the work done by the force $f(x,y,z) = (x^2y, xy^2, z^2)$ in moving an object from $(0,0,0)$ to $(1,2,3)$ along the curve $C: [0,1] \to \mathbb{R}^3 : C(t) = (t, 2t^2, 3t^3)$. 10%

3. (i) Suppose $0 < c < d$. Evaluate $\lim_{n\to\infty} \sqrt[n]{c^n + d^n}$. 5%

   (ii) Evaluate $\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}\,dt$. 10%

4. (i) Evaluate $\iint_Q |\cos(x+y)|\,d(x,y)$, where $Q = [0,\frac{\pi}{2}] \times [0,\frac{\pi}{2}]$. 10%

   (ii) Use Green's Theorem to evaluate

$$\int_C (e^x - 3xy)\,dx + (\sin y + 6x^2)\,dy,$$

   where $C: [0,2\pi] \to \mathbb{R}^2 : C(t) = (\cos t, 2\sin t)$. 10%

5. Define $f: \mathbb{R} \to \mathbb{R} : f(x) = \begin{cases} (x^2)^x, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$ Is f continuous at 0? 10%

6. Suppose $0 < p < q$ and $\int_0^\infty \frac{1}{x^p + x^q}\,dx$ converges. Show that $0 < p < 1 < q$. 10%

7. Suppose the temperature distribution on a circular wire ring is a continuous function. Prove that there is a pair of diametrically opposite points at which the temperatures are equal. (Hint : use the intermediate value theorem). 10%

8. Suppose $f: [0,1] \to \mathbb{R}$ has a continuous derivative satisfying $0 < f'(x) \le 1$ for all $x \in [0,1]$. Also, suppose $f(0) = 0$. Prove that $\left[\int_0^1 f(t)\,dt\right]^2 \ge \int_0^1 (f(t))^3\,dt$. (Hint : consider the function $F: [0,1] \to \mathbb{R} : F(x) = (\int_0^x f(t)\,dt)^2 - \int_0^x (f(t))^3\,dt$.) 10%