1. Please find the general solutions of the following equations (5% for each one):
   a. $3xy' + y + x^2y^4 = 0$
   b. $(x-2)^2y'' + 2(x-2)y' - 6y = 0$
   c. $xy' + y = y\ln(xy)$ where $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$

2. Please solve the following simultaneous differential equations (5%):

$$\begin{cases} x' + 2x + y' + 4y = 1 \\ x' + x + y' + 5y = 2 \end{cases}$$ where $x' = \frac{dx}{dt}$ and $y' = \frac{dy}{dt}$

3. (10%) Please solve the following partial differential equation: $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$

with initial condition at t=0, $T(0,x) = T_o$, and boundary conditions at x=0, $T(t,0)=T_o$, at $x=\infty$, $T(t,\infty) = 0$.

4.(15%) Please write down the definition of Laplace transformation, $L\{f(t)\} = ?$ Under what conditions does Laplace transform integral converge uniformly ? Please solve the following simultaneous equations with the method of Laplace transform:

$$\begin{cases} y' + 2y + 6\int_0^t z\,dt = -2u(t) \\ y' + z' + z = 0 \end{cases}$$ where $y' = \frac{dy}{dt}$, $z' = \frac{dz}{dt}$

with y(0)=-5 and z(0)= 6.

5.(10%) The following simultaneous equations can be expressed in matrix form as [A][x]=[B]:

$$\begin{cases} 5x + 3y - 3z = -1 \\ 3x + 2y - 2z = -1 \\ 2x - y + 2z = 8 \end{cases}$$

a. Write down the expression of A, B, and X;
b. Determine the eigenvalues of matrix A;
c. Determine the inversion of matrix A, that is, A^{-1};
d. Determine the solution of the equations.

6.(10%) The integral expression, derived from control volume approach, for the conservation of mass is

$$\iint \rho(\vec{V}.\vec{n})\,dA + \frac{\partial}{\partial t}\iiint \rho\,dv = 0$$

show that the equation can be expressed in the differential form as

$$\frac{\partial \rho}{\partial t} + \vec{V}.\nabla\rho + \rho\nabla.\vec{V} = 0$$

7.(10%) In the predator-prey model of Volterra, we are concerned with only two species: the predators and their prey. The populations of predator x(t) and prey y(t) can be modeled as:

$$\begin{cases} \frac{dx}{dt} = ax - bxy \\ \frac{dy}{dt} = -cy + dxy \end{cases}$$

where t is time and a, b, c, d are constants. Determine the relationship between the population of predator and their prey at any time.

8.(10%) Einstein (1905) has developed the equation for the spread of particle position due to Brownian motion. The motion for each individual particle according to Newton's

second law is : $m\frac{du}{dt}=-fu+A$

where m is the particle mass, u = dx/dt is the particle velocity,x is the particle position, t is time, and f and A are constants. The average of position for all particles is zero and is

defined as : $\overline{x}=\int x*n(x)\,dx$

where n(x) is the distribution function for the fraction of particle at location x. Prove that

the spread of particle position, which is defined as : $\overline{x^2}=\int x^2 n(x)\,dx$

is given as the following equation: $\overline{x^2}=\frac{\overline{u^2}}{k}\left[t+\frac{1}{k}(e^{-kt}-1)\right]$

where k = f/m.

9.(15%) For the general first order differential equation dy/dx = f(x,y), the value of y can be determined by Runge-Kutta method from the given value y(x=0). The formula is:

$$y_{n+1} = y_n + (a+2*b+2*c+d)/6,$$

where a = h*f(x_n, y_n); b = h*f(x_n+0.5*h, y_n+0.5*a); c = h*f(x_n+0.5*h, y_n+0.5*b); d = h*f(x_n+h, y_n+c), and h is the increment of one step in x. Write down the complete calculation formulas and steps to compute the value of y at x=1, given the value of y at x=0, for the equation: $y''' + 2\,y\,y'' + 5\,x^2\,y\,y' - 3x^2\,y^3 = 6$.