※ 考生請注意：本試題 □可 ☑不可 使用計算機

1. (15%) Solve $\left[(x+3)D^2-(2x+7)D+2\right]y=(x+3)^2e^x$

2. (20%) Solve $xr+2p=(9x+6)e^{3x+2y}$, where $r=\partial^2 z/\partial x^2$ and $p=\partial z/\partial x$

3. (15%) Evaluate the following integral

$$\int_{-1}^{1}\frac{dt}{t^2+i}$$

4. (25%) Mark each of the following statements True (T) or False (F). (Need NOT to give reasons.)

   (a) A real square matrix may have complex eigenvalues and complex eigenvectors.

   (b) Let M be a symmetric matrix. If M is invertible, then M^{-1} is also a symmetric matrix.

   (c) Let M be a real square matrix of size n. If $\|M\mathbf{x}\|^2=\|\mathbf{x}\|^2$ for all $\mathbf{x}\in\mathbb{R}^n$, then M is an orthogonal matrix, $M^TM=I_n$.

   (d) Let M be an $m\times n$ matrix, $m\neq n$. We have $\text{rank}(M^TM)=\text{rank}(MM^T)$.

   (e) Let M be an $m\times n$ matrix, $m\neq n$. We have $\text{nullity}(M^TM)=\text{nullity}(MM^T)$.

5. (15%) Suppose that A is a square matrix of size n, and $\lambda_1,\ldots,\lambda_k$ are distinct eigenvalues of A, with the corresponding multiplicity $m_1,\ldots,m_k$, respectively, where $m_1+\cdots+m_k=n$. Prove the determinant of A is

$$\det(A)=\lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_k^{m_k}.$$

6. (10%) Let I_m and I_n be identity matrices of sizes m and n, respectively, where we assume $m>n$. Can you find an $m\times n$ matrix A and an $n\times m$ matrix B such that $AB=I_m$ and $BA=I_n$? (Explain your answer.)