

※ 考生請注意：本試題不可使用計算機。請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. Solve

(a) (10%) $xy'' + xy' - y = 0 \quad (0 < x < \infty)$

(b) (10%) $r^2 R'' + rR' - \kappa^2 R = 0 \quad (\kappa \geq 0)$

2. (15%) Find the eigenvalues and associated eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0 \quad (0 < x < L)$$

$$y'(0) = 0, \quad y(L) = 0$$

3. As we know, equation

$$\frac{d}{dx} \left(x^a \frac{dy}{dx} \right) + bx^c y = 0 \quad (\text{I})$$

where a, b, c are real numbers, can be transformed to a Bessel equation of order ν , i.e.,

$$t^2 \frac{d^2 u}{dt^2} + t \frac{du}{dt} + (t^2 - \nu^2) u = 0 \quad (\text{II})$$

by transforming with $\alpha\sqrt{b}x^{1/\alpha} = t$ and $x^{-\nu/\alpha}y = u$.

Now, consider the equation

$$x^3 y'' - x^2 y' + y = 0 \quad (\text{III})$$

(a) (10%) Is it really of the form (I), if so, what are the associated a, b, c values of (III)?

(b) (5%) Find the general solution of (III).

4. (20%) Let A be an $n \times n$ matrix.

(a) Suppose that A is non-invertible. Can you find an $n \times n$ matrix B such that BA is an invertible matrix? (Explain your answer.)

(b) Suppose that A is invertible. Can you find an $n \times n$ matrix B such that BA is a non-invertible matrix? (Explain your answer.)

5. Let T be a transformation from $M^{4 \times 4}$ to \mathbb{R} , defined by $T(M) = \text{tr}(M)$, the trace of M .

(a) (10%) Is T a linear transformation?

(b) (5%) Is T a one-to-one transformation?

(c) (5%) Is T an onto transformation?

(d) (10%) Find the rank and nullity of T .