

※ 考生請注意：本試題不可使用計算機。請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (10%) An f -graph is a directed graph where each vertex has outdegree 1.

An r -graph is a directed graph with no multiple edges. That is, for all vertices u, v , there can be at most one edge from u to v .

(f : function, r : relation)

(a) Is every f -graph also an r -graph? Give an example of 3 nodes.

(b) Is every r -graph also an f -graph? Give an example of 3 nodes.

2. (10%) Define a **set** for the outcome of the following scenarios.

Scenario (a): A professor chooses two students to work together on a problem.

Scenario (b): A professor asks two questions in sequence, and each time calls on a student to answer.

Note: We can describe the set of all elements of S that have property p as $\{x \in S \mid x \text{ has property } p\}$.

3. (15%) Suppose there are n pigeons, and these birds make homes in p holes.

(a) If $n=12$ and $p=10$, can every pigeon have its own hole? (2%)

(b) If $n=22$ and $p=10$, what must happen? (3%)

(c) State a general principle involving n and p . (8%)

(d) Use (c) to explain your answer to (a) and (b). (2%)

4. (15%) Define a sequence of shapes as follows:

Base case: $K(1)$ is an equilateral triangle.

Recursive Case: For $n > 1$, $K(n)$ is formed by replacing each line segment of $K(n-1)$ with the shape

such that the central vertex points outwards.

(a) Draw $K(3)$.

(b) How many line segments contains in $K(n)$.

(c) Prove (b) by induction.

5. (10%) Let m and n denote positive integers. Suppose a function Q is defined recursively as follows:

$$Q(m, n) = \begin{cases} 0 & \text{if } m \leq n \\ Q(m - n, n) + 1 & \text{if } n \leq m \end{cases}$$

(a) Find the value of $Q(2, 3)$ and $Q(14, 3)$.

(b) What does this function do? Find $Q(5861, 7)$.

6. (20%) There are four players in the game of bridge who are called North, South, East, and West. Each player is dealt 13 cards and the distribution of the cards is called a bridge hand. (You can leave answers in factorial notation.)
- Find the number x of possible bridge hands.
 - Find the number y of possible bridge hands if one player is to be dealt all four Aces.
 - Find the number z of possible bridge hands if West is to be dealt eight spades and North the other five spades.
 - Find the number u of possible bridge hands if North and South have, together, all four Aces.

7. (20%) A network is a collection of objects and their relationships to one another. The “objects” that are connected are called nodes, vertices, entities, and in some contexts people. The connections between the vertices are called edges, ties, and links. The following figure is an example of social network, where each node represents a person and each edge stands for the “friend” relationship between two persons. In graph theory and network analysis, we define several **Centrality** measures to identify important nodes in a graph. In this question, we introduce three kinds of Centrality measures:

- **Degree** centrality is a simple count of the total number of connections linked to a node.
- **Betweenness** centrality is a measure of how often a given node lies on the shortest path between two other nodes.
- **Closeness** centrality is the average distance between a node and every other nodes in the network.

- (6%) Please indicate who have highest **Degree** (i.e. the most important person based on Degree centrality definition), and explain your answer in a short paragraph.
- (7%) Please indicate who have highest **Betweenness** (i.e. the most important node based on Betweenness centrality definition), and explain your answer in a short paragraph.
- (7%) Please indicate who have lowest **Closeness** (i.e. the most important ones based on Closeness centrality definition), and explain your answer in a short paragraph.

