國立成功大學106學年度碩士班招生考試試題

系　所：電腦與通信工程研究所

考試科目：通信數學

考試日期：0214，節次：3

※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (20%) The joint probability density function (pdf) of random variables (RVs) X and Y is given by $f_{X,Y}(x,y) = ce^{-x}$ if $x \geq 0$ and $|y| < x$; $f_{X,Y}(x,y) = 0$ if elsewhere. Note that c is a constant (but you need to determine its value by yourself).
   (a) Determine the conditional pdf $f_{Y|X}(y|x)$ for $x \geq 0$.
   (b) Assuming $x \geq 0$, find the conditional expectation $\mathrm{E}(Y|X = x)$ and the conditional variance $\mathrm{Var}(Y|X = x)$.

2. (15%) Consider n jointly distributed RVs, $X_1, \ldots, X_n$. Let $a_1, \ldots, a_n$ be constants.
   (a) Express $\mathrm{Var}(\sum_{i=1}^{n} a_i X_i)$ in terms of $\mathrm{Var}(X_i)$ and $\mathrm{Cov}(X_i, X_j)$, which denotes the covariance of X_i and X_j.
   (b) [continued from part (a)] If, in addition, X_i's are mutually independent, determine $\mathrm{Var}(\sum_{i=1}^{n} a_i X_i)$.

3. (15%) At even time instants, a robot moves either $+\delta$ centimeters (cms) or $-\delta$ cms in the x-direction if the outcome of a coin flip is a head or a tail, respectively. At odd time instants, a robot moves either $+\delta$ cms or $-\delta$ cms in the y-direction if the outcome of a coin flip is a head or a tail, respectively. Assuming that the robot begins at the origin, let X and Y be the x- and y-coordinates, respectively, of the location of the robot after $2n$ time instants. We assume that the outcome of each coin flip is *independent* of the others; the probability of getting a head in each flip is denoted as q. Find the joint probability mass function (PMF) of X and Y, $p_{X,Y}(x,y)$.
   Note: The first time that the robot moves is along the y-direction.
   Hints: First, find the marginal PMF of X, $p_X((-n+2k)\delta) = P(X = (-n+2k)\delta)$ where k is an integer. (Think about what the range of k is.) Similarly, find the marginal PMF of Y, $p_Y((-n+2m)\delta) = P(Y = (-n+2m)\delta)$ where m is an integer. (Also think about what the range of m is.)

4. (20%) Suppose that A and B are two $n \times n$ matrices. Choose the true statement(s) from the following.
   (a) A is an invertible matrix if and only if all eigenvalues of A are non-zero.
   (b) If the set of columns of A is an orthogonal set, then the set of rows of A is also orthogonal.
   (c) If $AB = BA$, then A and B have the same row space.
   (d) If $AB = BA$, then A and B have the same column space.

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5. (15%) Suppose that V is a vector space, and T is a linear operator on V. Choose the true statement(s) from the following.
   (a) If T has an eigenvector $\mathbf{x}$, then $\mathbf{x}$ is also an eigenvector of $2T$.
   (b) If T has an eigenvalue λ, then λ is also an eigenvalue of $2T$.
   (c) T^2 is also a linear operator on V.

6. (15%) Suppose that M is an invertible matrix. Prove that the matrix $(M + M^{-1})$ is also an invertible matrix.