※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (30%) Find the eigenvalues and corresponding eigenfunctions of the regular Sturm-Liouville problem

$$\begin{cases} x^2 y'' + 3xy' + \lambda y = 0, & \text{for } 1 < x < e \\ y(1) = 0 \text{ and } y(e) = 0 \end{cases}$$

Also give an orthogonality relation for the eigenfunctions.
(Skip the $1 - \lambda = 0$ and $1 - \lambda > 0$ cases)

2. (20%) Evaluate

$$\int_\gamma \frac{|z|e^z}{z^2} dz$$

where γ is the circle with radius 2 and center 0.

3. (20%) Suppose that A and B are two $n \times n$ matrices. Choose the true statement(s) from the following.

(a) If the rows of A are linearly dependent, then the columns of A are also linearly dependent.

(b) A is an invertible matrix if and only if all eigenvalues of A are non-zero.

(c) If $AB = BA$, then A and B have the same row space.

(d) If $AB = BA$, then A and B have the same column space.

4. (15%) Suppose that V is a vector space, and T is a linear operator on V. Choose the true statement(s) from the following.

(a) If T has an eigenvector $\mathbf{x}$, then $\mathbf{x}$ is also an eigenvector of $3T$.

(b) If T has an eigenvalue λ, then λ is also an eigenvalue of $3T$.

(c) T^2 is also a linear operator on V.

5. (15%) Suppose that M is an invertible matrix. Prove that the matrix $(M + M^{-1})$ is also an invertible matrix.