※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (20%) A transmitter sends information over a noisy channel by the following repetition coding scheme. The information bit takes on 0 or 1 with equal probability, and information bits are mutually independent. The transmitter repeats every information bit five times. In other words, for each information bit, the transmitter sends out 5 repeated coded bits. Specifically, if the information bit is 1, then the 5 coded bits that are sent to channel are 11111. If the information bit is 0, then the 5 coded bits that are sent to channel are 00000. We call each such a group of five coded bits a codeword. The channel changes a coded bit to its complement (i.e., $0 \to 1$ or $1 \to 0$) with probability p, and it does so independently of its treatment of other coded bits. The receiver takes a majority vote of the five received coded bits (that belong to a codeword) to guess at the transmitted information bit.
   (a) Find the probability that the receiver makes the wrong decision on the transmitted information bit.
   (b) Any advantages of this scheme over the scheme without repetition? Any disadvantages of this scheme over the scheme without repetition? Please comment briefly.

   Hints:
   i) Example: Let us assume that the information bit is 0 in a particular realization. Then the codeword transmitted is 00000 (i.e., this is the input to the channel). Let us assume that, in a particular channel realization, the corresponding received coded codeword is 00111 (i.e., this is the output of the channel). The outcome of the majority vote (which is the decision) would be 1 since the majority of the received coded bits is 1. In this case, the decision is wrong since the transmitted information bit is 0.
   ii) P(making a wrong decision) = P({info. bit=0 and decision=1} or {info. bit=1 and decision=0}).

2. (30%) The joint probability density function (pdf) of random variables X and Y is given by $f_{X,Y}(x,y) = \frac{1}{\pi r^2}$ if $x^2+y^2 \le r^2$; $f_{X,Y}(x,y) = 0$ if elsewhere. Note that $r>0$ is a constant.
   (a) Determine the conditional pdf $f_{Y|X}(y|x)$ for $-r \le x \le r$. Name this conditional distribution.
   (b) Find the conditional expectation $\mathrm{E}(Y|X=x)$ for $-r \le x \le r$.
   (c) Find the conditional variance $\mathrm{Var}(Y|X=x)$ for $-r \le x \le r$.

(Turn Back)

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3. (20%) Choose the true statement(s) from the following.

   (a) If an $n \times n$ matrix A has n distinct non-zero eigenvalues, then the rank of A is n.

   (b) If all eigenvalues of an $n \times n$ matrix A are zero. then the rank of A is 0.

   (c) Let T be a linear transformation (operator) on a vector space V. Then $T + \mathbf{v}_0$ is also a linear operator on V, where $\mathbf{v}_0$ is a constant vector in V.

   (d) Suppose that the matrices A, B, and C satisfy $AB = AC$. If A is an invertible square matrix, then we have $B = C$.

4. Suppose that M is a 4×5 matrix with rank 4.

   (a) (10%) Is it possible that $M^T M$ an invertible matrix? (Give your reasons.)

   (b) (20%) Let I be the 5×5 identity matrix. Is $(I + M^T M)$ an invertible matrix? (Explain you answer.)