

※ 考生請注意：本試題不可使用計算機。請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (20%) Solve the following ODEs in terms of Bessel functions, or elementary functions, if possible.

$$xy'' + (1 + 6x^2)y' + x(2 + 9x^2)y = 0$$

2. (10%) Let $f(z) = \frac{1}{(z^2 + z)(z + 2)^3}$

Compute the integral of $f(z)$ on the circles of center 1 and radii $\frac{1}{2}$, and $\frac{3}{2}$, respectively.

3. (20%) Consider the ODE $\frac{d^2y}{dx^2} + p^2y = h(x)$, $0 < x < \pi$, with the boundary conditions $y(0) = 0 = y(\pi)$; $y = y(x)$ is unknown and p is real, $p > 0$. Find the solution $y(x)$ of this boundary-value problem as an expansion in suitable Fourier series. For what values of p does the problem have a solution?

4. (20%) Choose the true statement(s) from the following.

- (a) If an $n \times n$ matrix A has n distinct non-zero eigenvalues, then the rank of A is n .
- (b) If all eigenvalues of an $n \times n$ matrix A are zero, then the rank of A is 0.
- (c) Suppose that the matrices A , B , and C satisfy $AB = AC$. If A is an invertible square matrix, then we have $B = C$.
- (d) Let T be a linear transformation (operator) on a vector space V . Then $T + 3I$ is also a linear operator on V , where I denotes the identity operator.

5. Suppose that M is a 3×4 matrix with rank 3.

- (a) (10%) Is it possible that $M^T M$ an invertible matrix? (Give your reasons.)
- (b) (20%) Let I be the 4×4 identity matrix. Is $(I + M^T M)$ an invertible matrix? (Explain your answer.)