1. (16 %) Solve the following differential equation

$$y'' + x(y')^2 = 0$$

2. (16 %) Solve the following differential equation

$$2y^2y'' + 2y(y')^2 = 1$$

3. (18 %) Two linearly independent solutions of Bessel's equation of order one-half

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0, \quad x > 0$$

are $x^{-1/2}\sin x$ and $x^{-1/2}\cos x$. Find the general solution of

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 3x^{3/2}\sin x, \quad x > 0$$

by using the Method of Variation of Parameter.

4. (14 %) Let $\mathbf{J}_n$ be an $n \times n$ matrix each of whose entries is 1. Please show that $(\mathbf{I} - \mathbf{J}_n)^{-1} = \mathbf{I} - \frac{1}{n-1}\mathbf{J}_n$.

5. (10%, 6%) Let $\boldsymbol{A}$ be an $m \times n$ matrix. (a) Show that $\boldsymbol{BA}$ and $\boldsymbol{A}$ have the same nullspace, if $\boldsymbol{B}$ is a nonsingular $m \times m$ matrix. (b) Show that $\boldsymbol{AC}$ and $\boldsymbol{A}$ have the same rank, if $\boldsymbol{C}$ is a nonsingular $n \times n$ matrix.

6. (10%, 10%) (a) Find a orthonormal basis for the subspace spanned by $\{1, \cos x\}$ in the vector space C[0, π] with the inner product defined by $\langle f, g \rangle = \frac{1}{\pi}\int_0^{\pi} f(x)g(x)dx$. (b) Find the best least squares approximation to x on the interval [0, π] by a linear combination of 1 and cos x.