

本試題是否可以使用計算機: 可使用 · 不可使用 (請命題老師勾選)

1. (20%) Solve $x \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 0$ by setting $x = t^\alpha$ and choosing α suitably.
2. (15%) If n is a positive integer, then the equation $x \frac{d^2 y}{dx^2} + 2n \frac{dy}{dx} + kxy = 0$ is satisfied by $y = \left(\frac{1}{x} D\right)^n z$, D is a differential operator. Where z is a solution of the equation $\frac{d^2 z}{dx^2} + kz = 0$. Thus find the general solution of $x \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4xy = 0$.
3. (15%) Find the general solution of the following system of equations
- $$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = z, \quad \frac{dz}{dt} = x$$
4. (6%, 6%, 8%) Consider the vector space $C[0, 1]$ with inner product $\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)dx$ and norm $\|f\| = (\langle f, f \rangle)^{1/2}$. Let S be the subspace spanned by the vectors 1 and $2x - 1$. (a) Are the vectors 1 and $2x - 1$ orthogonal? Why? (b) Determine $\|1\|$ and $\|2x - 1\|$. (c) Find the best least squares approximation to \sqrt{x} by a function from the subspace S .
5. (14%) A linear transformation $L: V \rightarrow W$ is said to be one-on-one if $L(v_1) = L(v_2)$ implies that $v_1 = v_2$. Show that L is one-on-one if and only if $\ker(L) = \{0_V\}$.
6. (16%) Let A be a real symmetric $n \times n$ matrix. Show that A is positive definite if and only if all its eigenvalues are positive.