1. Consider a population of size n, $n \geq 0$, of a certain species in which individuals do not reproduce. However, new individuals immigrate into the population at a Poisson rate of γ. If the lifetime of an individual in the population is exponential with mean $1/\mu$, find π_i , $i \geq 0$, the long-run probability that the population size is i. (10%)

2. From the set of integers {1, 2, 3, . . . , 100000} a number is selected at random. What is the probability that the sum of its digits is 8? (10%)

3. Adam and three of his friends are playing bridge. (a) If, holding a certain hand, Adam announces that he has a king, what is the probability that he has at least one more king? (b) If, for some other hand, Adam announces that he has the king of diamonds, what is the probability that he has at least one more king? (15%)

4. Suppose that n random integers are selected from $\{1, 2, \ldots, N\}$ with replacement. What is the expected value of the largest number selected? Show that for large N the answer is approximately $nN/(n+1)$. (15%)

(背面仍有題目,請繼續作答)

本試題是否可以使用計算機：☑可使用，□不可使用　（請命題老師勾選）

5. Let $\mathbf{x} \in R^n$. Show that $\|\mathbf{x}\|_1 \le n\|\mathbf{x}\|_\infty$ and $\|\mathbf{x}\|_2 \le \sqrt{n}\|\mathbf{x}\|_\infty$. (2×5%)

6. Let $\mathbf{u}$ be a unit vector in R^n and let $A = I - 2\mathbf{u}\mathbf{u}^T$. Determine A^{-1}. (10%)

7. (a) Is the matrix $A = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \beta \end{bmatrix}$ defective? Why? (b) Find e^{At}. (4%, 6%)

8. Suppose $A\mathbf{x} = \lambda\mathbf{x}$, $A \in R^{n\times n}$, and $\mathbf{x} \ne 0$. Is e^λ an eigenvalue of e^A? Why? (10%)

9. Suppose $A\mathbf{x} = \lambda\mathbf{x}$, $A \in C^{n\times n}$, $\mathbf{x} \ne 0$, and A is a skew Hermitian matrix. Is λ real or purely imaginary? Why? (10%)