※ 考生請注意：本試題 □可 ☑不可 使用計算機

1. (10%) Solve $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

2. (20%) Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{\cos(y)}{\sqrt{y^2+1}}dy$$
Given that the Hankel function $H_0^{(1)}(jw)$ is defined as
$$H_0^{(1)}(jw) = \frac{1}{\pi j} \int_1^{\infty} \frac{e^{-wx}}{\sqrt{x^2-1}}dx$$

3. (20%) For $f(x)$ in $L^2(0, \ell)$, find $u(x, t)$ satisfying
$$u_t = \kappa u_{xx} \qquad 0 < x < \ell, \, t > 0$$
$$u(x, 0) = f(x) \qquad 0 < x < \ell$$
$$u(0, t) = u(\ell, t) = 0 \qquad t > 0$$

4. (30%) Let A be an 5×5 matrix with eigenvalues $1, -1, 2, -2, 3$.

   (a) Find the determinant of A^2.

   (b) Is A an invertible matrix? (State the reason.)

   (c) Is A similar to a diagonal matrix? (State the reason.)

   (d) Find the trace of A. (The trace of A is the sum of the diagonal elements of A.)

   (e) Does the limit $\lim_{n\to\infty} A^n$ exist? Find this limit if it exists.

5. (20%) Consider a vector space of polynomials, defined by
$$V = \{p(t) = c_0 + c_1t + c_2t^2 \mid 0 \le t \le 1, \, c_0, c_1, c_2 \text{ are real numbers.}\}$$

Let T be a transformation with domain V, and $T(p(t)) = tp'(t) + 6\int_0^1 p(t)dt$, where $p'(t)$ is the derivative of $p(t)$.

   (a) Is T a linear transformation?

   (b) Find the eigenvalues of T and give a corresponding eigenfunction for each eigenvalue.