系所組別： 電腦與通信工程研究所丙組
考試科目： 電磁數學

※ 考生請注意：本試題 □可 ☑不可 使用計算機

1. (15%) Solve $(y')^4 + xy' = y - x$.

2. (20%) Solve (by separation of variables) for the steady-state temperature $u(r, z)$ in a semi-infinite rod of radius a if $u = 100$ at the end of the rod and $u = 0$ on its lateral surface; that is

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + u_{zz} = 0 \quad \text{in} \quad r < a, \; 0 < z < \infty$$
$$u(r, 0) = 100, \quad u(a, z) = 0.$$

3. (15%) Find the derivative $f'(z)$, where it exists, and state where f is analytic.

$$f = \frac{x + iy}{x^2 + y^2}$$

4. (25%) Mark each of the following statements True (T) or False (F). (Need NOT to give reasons.)

   (a) If all the eigenvalues of a square matrix A are zero, then A must be a zero matrix, $A = O$.

   (b) Let E_λ be an eigenspace of a square matrix A. Then all vectors in E_λ are eigenvectors of A.

   (c) Let A be an $m \times n$ matrix, $m \neq n$. Then $\text{rank}(A^T A) = \text{rank}(AA^T)$.

   (d) Let A be an $m \times n$ matrix, $m \neq n$. Then $\text{nullity}(A^T A) = \text{nullity}(AA^T)$.

   (e) A linear transformation T is one-to-one *if and only if* $N(T) = \{\mathbf{0}\}$, where $N(T)$ is the null space of T and $\mathbf{0}$ denotes the zero vector in the domain of T.

5. (10%) Let V be a vector space, and W_1 and W_2 be two subspaces of V. Is it possible that the intersection of W_1 and W_2, $W_1 \cap W_2 = \phi$, where ϕ denotes the empty set. (Explain your answer.)

6. Let A be a real symmetric matrix of size n, with the spectral decomposition as

$$A = \lambda_1 P^{(1)} + \lambda_2 P^{(2)} + \cdots + \lambda_K P^{(K)},$$

where each $P^{(k)}$, $1 \leq k \leq K$, is an orthogonal projection matrix. The range of $P^{(k)}$ is the kth eigenspace of A.

   (a) (5%) Find A^3. (Express A^3 by $\lambda_1, \ldots, \lambda_K$ and $P^{(1)}, \ldots, P^{(K)}$.)

   (b) (10%) Under what condition will A be invertible? If A is invertible, find A^{-1}. (Express A^{-1} by $\lambda_1, \ldots, \lambda_K$ and $P^{(1)}, \ldots, P^{(K)}$.)