系所組別： 電腦與通信工程研究所乙組
考試科目： 通信數學

※ 考生請注意：本試題 □可 ☑不可 使用計算機

1. (15%) If X is a random variable with expected value μ and variance σ^2, then for any $t > 0$, prove that $P(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$.

2. (15%) From families with three children, *a child* is selected at random and found to be a girl. What is the probability that she has an older sister? Assume that in a three child family all sex distributions are equally probable.

3. (10%) The distribution function of a random variable X is given by $F(x) = \alpha + \beta \arctan(x/2)$, $-\infty < x < \infty$. Determine the constants α and β and the density function of X.

4. (10%) There are 25 students in a probability class. What is the expected number of the days of the year that are birthdays of at least two students? Assume that the birthrates are constant throughout the year and that each year has 365 days.

5. (25%) Mark each of the following statements True (T) or False (F). (Need NOT to give reasons.)

   (a) Let E_λ be an eigenspace of a square matrix A. Then all vectors in E_λ are eigenvectors of A.

   (b) If all the eigenvalues of a square matrix A are zero, then A must be a zero matrix, $A = O$.

   (c) Let A be an $m \times n$ matrix, $m \neq n$. Then $\text{nullity}(A^T A) = \text{nullity}(AA^T)$.

   (d) Let A be an $m \times n$ matrix, $m \neq n$. Then $\text{rank}(A^T A) = \text{rank}(AA^T)$.

   (e) A linear transformation T is one-to-one *if and only if* $N(T) = \{\mathbf{0}\}$, where $N(T)$ is the null space of T and $\mathbf{0}$ denotes the zero vector in the domain of T.

6. (10%) Let V be a vector space, and W_1 and W_2 be two subspaces of V. Is it possible that the intersection of W_1 and W_2, $W_1 \cap W_2 = \phi$, where ϕ denotes the empty set. (Explain your answer.)

7. Let A be a real symmetric matrix of size n, with the spectral decomposition as

$$A = \lambda_1 P^{(1)} + \lambda_2 P^{(2)} + \cdots + \lambda_K P^{(K)},$$

where each $P^{(k)}$, $1 \leq k \leq K$, is an orthogonal projection matrix. The range of $P^{(k)}$ is the kth eigenspace of A.

   (a) (5%) Find A^2. (Express A^2 by $\lambda_1, \ldots, \lambda_K$ and $P^{(1)}, \ldots, P^{(K)}$.)

   (b) (10%) Under what condition will A be invertible? If A is invertible, find A^{-1}. (Express A^{-1} by $\lambda_1, \ldots, \lambda_K$ and $P^{(1)}, \ldots, P^{(K)}$.)