

※ 考生請注意：本試題不可使用計算機。請於答案卷(卡)作答，於本試題紙上作答者，不予計分。只有答案，沒有計算過程或說明原因，也不予計分。考卷含線性代數及離散數學

一、線性代數 (50%)

1. (15%) Given  $A$  and  $p$ , if  $\lim_{n \rightarrow \infty} A^n p = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ . Please find  $a, b, c, d$ .

$$A = \begin{bmatrix} 1 & 2/5 & 1/10 & 0 \\ 0 & 3/10 & 1/2 & 0 \\ 0 & 0 & 1/5 & 0 \\ 0 & 3/10 & 1/5 & 1 \end{bmatrix}, P = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

2. (10%) Let  $V$  be the vector space  $\mathbb{R}^3$ . Determine if  $U$  is a subspace of  $V$  if  $U$  contains the following vectors.

(a)  $U = \{(x, y, z) : x < 0\}$

(b)  $U = \{(x, y, z) : x + y + z = 3\}$

(c)  $U = \{(x, y, z) : x = 2z\}$

3. (10%) Let  $U = \mathbb{R}^n$  and  $V = \{(x_1, x_2, \dots, x_n) \in U \mid x_1 + x_2 + \dots + x_n = 0\}$ , please find a basis of  $V$  over  $\mathbb{R}$

4. (15%) Let  $T: P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$  be a linear operator that is defined according to

$$T(g(x)) = g(x) + x \frac{dg(x)}{dx} + \frac{dg(x)}{dx}, \text{ in which } P_2(\mathbb{R}) \text{ is a set of all polynomials with real-value coefficients}$$

and degree  $n, n=0,1,2$ . Please find the eigenvalues and the associated eigenvectors of the operator

$$5T + 2T^2 + T^{10}$$

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二. Discrete Mathematics (total: 50%)

5. In how many ways can 36 identical robots be assigned to five assembly lines with

- (a) at least four robots assigned to each line? (5%)  
 (b) at least four, but no more than ten, assigned to each line? (5%)

(no score if you give no details.)

6. Let  $D = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & 0 \\ & -1 & 2 & \ddots & \\ & & -1 & \ddots & -1 \\ 0 & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$ , i.e.,  $D(i, j) = \begin{cases} 2, & \text{if } i = j. \\ -1, & \text{if } |i - j| = 1. \\ 0, & \text{elsewhere.} \end{cases}$

- (a) Use recurrence relations to express determinant of  $D$ , i.e.,  $|D|$ . (5%)  
 (b) Find the general solution for  $|D_n|$ . (5%)  
 (c)  $|D_1| = 2, |D_2| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$ . Find  $|D_{100}|$ . (5%)

(no score if you give no details.)

7. Solve the following recurrence relations:  $6a_n - 5a_{n-1} + a_{n-2} = \sin(n\pi)$   
 with  $a_0 = 1, a_{-1} = a_{-2} = 0$ . (10%)(no score if you give no details.)

8. Let  $(Q, \oplus, \otimes)$  denote the field, where  $\oplus$  and  $\otimes$  are defined by  
 $a \oplus b = a + b - k$ ,  $a \otimes b = a + b - (ab/m)$ , for fixed elements  $k, m (\neq 0)$  of  $Q$ .  
 Determine the values for  $k$  and  $m$  in each of the following:

- (a) The zero element for the field is 5. (5%)  
 (b) The additive inverse of the element 8 is  $-7$ . (5%)  
 (c) The multiplicative inverse of 3 is  $1/6$ . (5%)

(no score if you give no details.)