※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (30%) Show that the *sample mean* and *sample variance*

$$\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i,\quad S^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2$$

are *unbiased* estimate of μ and σ^2 respectively, where X_i are random variables that have identical normal distribution with mean μ and variance σ^2.

2. (35%)

(a) Prove the *Markov's inequality*:

If X is a non-negative random variable, then for any value $a > 0$,

$$P[X \geq a] \leq \frac{E[X]}{a}.$$

(b) Derive the *Chebyshev's inequality* from Markov's inequality:

$$P\left[|X-\mu| \leq k\sigma\right] \geq 1-\frac{1}{k^2}.\quad (k \geq 1)$$

(c) A random variable X has a mean $\mu = 6$ and a variance $\sigma^2 = 9$. Estimate the probability $P(X \geq 12 \text{ or } X \leq 0)$.

3. (35%)

(a) If $\bar{x}$ is the mean of a random sample of size n from a population with known variance σ^2, find a (1-α)% confidence interval for the population mean μ in terms of $\bar{x}$, σ, n, and standard normal $z_{\alpha/2}$.

(b) A random sample of 100 car owners in a city shows that a car is driven on the average 23,500 km per year with a standard deviation of 3,900 km. Construct a 99% confidence interval for the average number of kilometers a car is driven annually in the city. (Given $z_{0.005} = 2.575$, $z_{0.01} = 2.329$, $z_{0.05} = 1.645$.)

(c) How large a sample is required if we want to be 99% confident that our estimate of the average number of kilometers driven by car owners in the city is off by less than 300 km?