1. (a) 10% Show that the following first-order nonlinear ODE

$$\frac{dy}{dx} + P(x)y + Q(x)y^2 = R(x), \quad Q(x) \neq 0,$$

can be transferred into a second-order linear ODE by the transformation

$$y = \frac{1}{Q}\left(\frac{1}{z}\frac{dz}{dx}\right).$$

(b) 5% Apply the above transformation to

$$\frac{dy}{dx} + \frac{2}{x}y + \frac{1}{2}x^3y^2 = \frac{1}{2x},$$

and find its equivalent second-order linear ODE.

2. (a) 10% A given periodic function $f(x)$ with period 2π is to be expanded within the interval $[0,\ 2\pi]$ as the following Fourier series

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$$

How do you determine A_0, A_n, and B_n in terms of the given function $f(x)$.

(b) 5% Repeat problem (a) by changing the period of $f(x)$ from 2π to T.

3. 30% Make use of three different methods to find the particular solution of the following ODE

$$\ddot{x} + 4\dot{x} + 3x = 10e^{-2t}.$$

4. (a) 10% Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

(b) 10% Use the above result to evaluate A^{100}.

5. In complex variable theory you have learned that a complex function $f(z) = u(x,y) + iv(x,y)$ is analytic, if and only if its real part $u(x,y)$ and imaginary part $v(x,y)$ satisfy the Cauchy-Riemann Condition:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(a) 10% Explain how to define an analytic function and how to use this definition to derive the Cauchy-Riemann Condition.

(b) 10% If $f(z)$ is an analytic function, show that both its real part $u(x,y)$ and imaginary part $v(x,y)$ satisfy the Laplace equation.