編號： 211　系所：奈米科技暨微系統工程研究所　科目：應用數學

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1. Hermite's differential equation has the form

$$y'' - 2xy' + 2\mu y = 0, \tag{1a}$$

where μ is a given number. The Hermite equation appears in the description of the wave-function of the harmonic oscillator. Any solution of this equation is called a Hermite function.

(a) (10%) Find two linearly independent solutions in the form of a power series

$$y(x) = \sum_{m=0}^{\infty} a_m x^m \tag{1b}$$

by verifying that a_m satisfies the following recurrence relation

$$a_{m+2} = -\frac{2(\mu - m)}{(m+1)(m+2)} a_m \tag{1c}$$

(b) (5%) Express the general solution in terms of the two free parameters a_0 and a_1.

(c) (10%) If $\mu = n$ is a non-negative integer, show that the Hermite function $y(x)$ is reduced to a polynomial denoted by $y = H_n(x)$. Find the Hermite polynomial $H_n(x)$ for $n = 0$, 1, 2, 3, 4.

2. In quantum mechanics, each physical observable is accompanied by an operator. The two basic quantum-mechanical operators are those corresponding to position (x, y, z) and momentum (p_x, p_y, p_z). One prescription for making the transition from classical to quantum mechanics is to perform the following replacement:

$$(x, y, z) \rightarrow (\hat{x}, \hat{y}, \hat{z}), \quad (p_x, p_y, p_z) \rightarrow (\hat{p}_x, \hat{p}_y, \hat{p}_z) = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}, \frac{\hbar}{i}\frac{\partial}{\partial y}, \frac{\hbar}{i}\frac{\partial}{\partial z}\right) \tag{2a}$$

where $\hat{x}$ is a multiplicative operator defined by $\hat{x}f = xf$, and similarly, $\hat{y}f = yf$ and $\hat{z}f = zf$; $\hbar$ is the Planck constant and i is the imaginary number $\sqrt{-1}$.

(a) (5%)Making the above substitution rules, find the corresponding quantum-mechanical operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ for the following components of classical angular momentum:

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x. \tag{2b}$$

(b) (15%)The commutator $[\hat{A}, \hat{B}]$ of two operator $\hat{A}$ and $\hat{B}$ is defined by

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{2c}$$

Using the results of (a), show

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y \tag{2d}$$

3. (15%) Continuing Problem 2, two operator $\hat{A}$ and $\hat{B}$ are said to commute if their commutator is zero, i.e., $[\hat{A}, \hat{B}] = 0$. Consider the following eigenvalue problems:

$$\hat{A}\psi_A = \lambda_A \psi_A, \quad \hat{B}\psi_B = \lambda_B \psi_B \tag{3a}$$

where ψ_A and λ_A are the eigenfunction and eigenvalue of $\hat{A}$; while ψ_B and λ_B are the

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eigenfunction and eigenvalue of $\hat{B}$. Show that if $\hat{A}$ and $\hat{B}$ are commuting, then they share the same eigenfunction, i.e., $\psi_A = \psi_B$. Hint: $[\hat{A}, \hat{B}] = 0$ means $\hat{A}\hat{B} = \hat{B}\hat{A}$.

4. A complex function $f(z)$ is said to be analytic in a domain R, if it is single-valued and differentiable at all points in R. The differentiability of $f(z)$ means that its derivative

$$f'(z) = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right] \tag{4a}$$

exists and is unique, i.e., its value does not depend upon the direction from which Δz approaches to zero. If we let $f(z) = u(x,y) + iv(x,y)$ and $\Delta z = \Delta x + i\Delta y$, then we have

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y), \tag{4b}$$

and the limit in Eq.(4a) is given by

$$\lim_{\Delta x, \Delta y \to 0}\left[\frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}\right]. \tag{4c}$$

(a) (5%) By assuming $\Delta z \to 0$ along the real axis, i.e., $\Delta z = \Delta x$ and $\Delta y = 0$, evaluate the limit in Eq.(4c).

(b) (5%) By assuming $\Delta z \to 0$ along the imaginary axis, i.e., $\Delta z = i\Delta y$ and $\Delta x = 0$, evaluate again the limit in Eq.(4c).

(c) (5%) The differentiability of $f(z)$ requires that the results obtained in (a) and (b) must be equal. Show that the required condition is just the Cauchy-Riemann relation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{4d}$$

(d) (5%) Since x and y are related to z and its complex conjugate z^* by

$$x = \frac{1}{2}\left(z + z^*\right), \quad y = \frac{1}{2i}\left(z - z^*\right), \tag{4e}$$

we may formally regard any function $f = u + iv$ as a function of z and z^*, rather than x and y. If we do this and examine $\partial f / \partial z^*$, we obtain

$$\frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*}. \tag{4f}$$

Show that if f is analytic, then f cannot be a function of z^*, i.e., $\partial f / \partial z^* = 0$. Hint: Apply Eq.(4e) to Eq.(4f) and use the Cauchy-Riemann condition (4d).

5. (20%) In the definition of an analytic function, one of the conditions imposed was that the function is singled-valued. A multi-valued function can still be treated as analytic if its principle value is taken. In polar form, a complex number z can be written as

$$z = re^{i\theta} = re^{i(\theta + 2n\pi)}, \quad n \in \mathbb{Z}, \quad -\pi < \theta \le \pi \tag{5a}$$

Taking the logarithm of both sides, we find

$$\text{Ln}(z) = \ln(r) + i(\theta + 2n\pi). \tag{5b}$$

Hence, the logarithm function $\text{Ln}(z)$ is multi-valued, depending on the value of n. The principle value of $\text{Ln}(z)$ is denoted by $\ln(z)$, which is evaluated at $n = 0$, i.e.,

$$\ln(z) = \ln(r) + i\theta, \quad -\pi < \theta \le \pi. \tag{5c}$$

According to the above definition, find the all possible values of $\mathrm{Ln}(-i)$ and its principle value $\ln(-i)$.