In this test, you are asked to solve the following set of differential equations:

$$\dot{x} \triangleq \frac{dx}{dt} = x - 2y, \quad x(0) = 1, \tag{1a}$$

$$\dot{y} \triangleq \frac{dy}{dt} = 2x + y, \quad y(0) = 1. \tag{1b}$$

by four different methods you have learned in the course of engineering mathematics.

1. The first method to solve Eqs.(1) is by direct substitution with coefficients to be determined. By eliminating the variable y between Eq.(1a) and Eq.(1b), show that the differential equation for x becomes

$$\ddot{x} + a\dot{x} + bx = 0 \tag{2}$$

   (1) (5%) Find the values of a and b.
   (2) (2%) Find the initial value $\dot{x}(0)$.
   (3) (8%) Assume that the solution to $x(t)$ has the following form

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}. \tag{3}$$

   Find the values of λ_1, λ_2, c_1, and c_2.
   (4) (5%) Find the solution to $y(t)$.

2. The second method to solve Eqs.(1) is by Laplace transform.
   (1) (10%) By taking Laplace transformation of Eq.(2), show that the Laplace transform $X(s)$ of $x(t)$ can be expressed in the following form:

$$X(s) = \frac{s + c}{s^2 + as + b} \tag{4}$$

   Find the values of a, b, and c.
   (2) (10%) By taking inverse Laplace transformation of Eq.(4) to find $x(t) = L^{-1}(X(s))$, show that the obtained $x(t)$ is the same as that in Eq.(3).

3. The third method to solve Eqs.(1) is by complex variable. Define a complex variable z as

$$z = x + yi \in \mathbb{C} \tag{5}$$

   where $x(t)$ and $y(t)$ satisfy Eqs.(1).
   (a) (8%) Combine Eq.(1a) and Eq.(1b) and express them in a single equation of z as

$$\frac{dz}{dt} = az + b. \tag{6}$$

   Find the values of a and b.
   (b) (2%) Find the initial value $z(0)$.
   (c) (10%) Solve Eq.(6) for $z(t)$ and show that the real part $x(t) = \mathrm{Re}(z(t))$ and the imaginary part $y(t) = \mathrm{Im}(z(t))$ are identical to the solutions found in problem (1).

4. The fourth method to solve Eqs.(1) is by linear algebra. Eqs.(1) can be recast into a matrix form as

$$\frac{dX}{dt} = AX, \quad X = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}. \tag{7}$$

In this problem and the next problem, you are asked to solve $X(t)$ by the eigenvalues and eigenvectors of the matrix A.

(a) (10%) By the following Taylor series expansion,

$$e^{At} = I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots, \tag{8}$$

show that the solution $X(t)$ to Eq.(7) can be expressed by

$$X(t) = e^{At}X(0). \tag{9}$$

(b) (5%) Find the two eigenvalues λ_1 and λ_2 of the matrix A in Eq.(7).

(c) (5%) Find the two eigenvectors V_1 and V_2 of the matrix A. Write down V_1 and V_2 in the following forms

$$V_1 = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}, \quad V_2 = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}. \tag{10}$$

(Take $v_{21} = v_{22} = 1$ and find the values of v_{11} and v_{12})

5. Continue the discussion of problem (4). Define the matrix V as

$$V = [V_1 \ \ V_2] = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}. \tag{11}$$

(a) (4%) Show that A can be expressed in terms of V as

$$A = V\Lambda V^{-1} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}^{-1} \tag{12}$$

(b) (3%) Show that the power of A can be evaluated by

$$A^n = V\Lambda^n V^{-1}, \quad n = 0,\ 1,\ 2,\ 3,\ \cdots \tag{13}$$

(c) (3%) Combine the results of Eq.(8) and Eq.(13) to show the result:

$$e^{At} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}^{-1}. \tag{14}$$

(d) (10%) Evaluate e^{At} in Eq.(14) by using the eigenvalues and eigenvectors found in problem (4) and then find the solution $X(t) = [x(t) \ y(t)]^T$ from Eq.(9). Verify that the obtained $x(t)$ and $y(t)$ are identical to those found in problem 1.