※ 考生請注意：本試題 ☑可 □不可 使用計算機

1. (a) (12%) Find the eigenvalues and eigenvectors of the matrix

$$A=\begin{bmatrix}10 & 3i\\ -3i & 2\end{bmatrix}$$

(b) (8%) Construct a 2×2 matrix U such that $U^{\dagger}AU=\Lambda$, where $U^{\dagger}=(U^{*})^{T}$ is the complex-conjugate transpose of U and Λ is a real diagonal matrix. (Hint: relate U to the unit-length eigenvectors of A and relate Λ to the eigenvalues of A.)

2. Suppose two functions $P(x,y)$ and $Q(x,y)$ are single-valued, finite and continuous inside and on the boundary C of some simply connected region R in the xy plane.
(a) (5%) Apply Green theorem on a plane to transform the following line integral

$$\oint_C (Pdx+Qdy)$$

into a double integral over the enclosed region R.
(b) (5%) Use the above Green theorem to show that the area enclosed by R can be expressed by

$$A=\iint_R dxdy=\frac{1}{2}\oint_C (xdy-ydx)$$

(c) (10%) Apply the above formula to calculate the area of the ellipse $x=a\cos\theta$, $y=b\sin\theta$.

3. (20%) A series electric circuit contains a resistance R, a capacitance C and a battery supplying a time-varying electromotive force $V(t)$. The charge q on the capacitor therefore obeys the equation:

$$R\frac{dq}{dt}+\frac{q}{C}=V(t)$$

Assuming that initially there is no charge on the capacitor, and given that $V(t)=V_0\sin\omega t$, find the charge on the capacitor as a function of time.

4. In this problem, you are asked to evaluate the following definite integral

$$I=\int_0^{2\pi}\frac{\cos 2\theta}{a^2+b^2-2ab\cos\theta}\,d\theta,\quad b>a>0$$

(a) (5%) Let $z=e^{i\theta}$ and then show

$$\cos\theta=\frac{1}{2}(z+z^{-1}),\quad \sin\theta=\frac{1}{2i}(z-z^{-1}),\quad \cos n\theta=\frac{1}{2}(z^{n}+z^{-n})$$

(b) (10%) In terms of the above relations, show that the integral I can be evaluated by the contour integral

$$I=\frac{i}{2ab}\oint_C\frac{z^4+1}{z^2(z-a/b)(z-b/a)}dz$$

and identify the contour C.
(c) (5%) Apply the residue theorem to find the value of I.

(背面仍有題目,請繼續作答)

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5. The wave equation describing the transverse vibration of a stretched membrane under tension T and having a uniform surface density ρ is

$$T\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\rho\frac{\partial^2 u}{\partial t^2}$$

(a) (15%) Find a separable solution to the above PDE for a membrane stretched on a frame of length a and width b (i.e., $u(0,y,t)=u(a,y,t)=0$ and $u(x,0,t)=u(x,b,t)=0$).

(b) (5%) Show that the natural angular frequencies of such a membrane are given by

$$\omega^2=\frac{\pi^2 T}{\rho}\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right).$$

where m and n are positive integers.