※ 考生請注意：本試題 ☑可 ☐不可 使用計算機

1. (a) (12%) Find the eigenvalues and a set of eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{bmatrix}$$

   (b) (8%) Verify that its eigenvectors are mutually orthogonal.

2. A vector field $\mathbf{V}$ is defined by

$$\mathbf{V} = 2xz\vec{\mathbf{i}} + 2yz^2\vec{\mathbf{j}} + (x^2 + 2y^2z - 1)\vec{\mathbf{k}}$$

   (a) (10%) Calculate $\nabla \times \mathbf{V}$.
   (b) (10%) Is it possible to find a function ϕ such that $\mathbf{V} = \nabla\phi$? If yes, find such a ϕ.

3. (20%) A radioactive isotope decays in such a way that the number of atoms present at a given time $N(t)$, obeys the equation

$$\frac{dN}{dt} = -\lambda N$$

   If there are initially N_0 atoms present, find $N(t)$ at later times.

4. In this problem, you are asked to evaluate the following definite integral

$$I = \int_0^{2\pi} \frac{\cos 2\theta}{a^2 + b^2 - 2ab\cos\theta}\, d\theta, \quad b > a > 0$$

   (a) (5%) Let $z = e^{i\theta}$ and then show

$$\cos\theta = \frac{1}{2}(z + z^{-1}), \quad \sin\theta = \frac{1}{2i}(z - z^{-1}), \quad \cos n\theta = \frac{1}{2}(z^n + z^{-n})$$

   (b) (10%) In terms of the above relations, show that the integral I can be evaluated by the contour integral

$$I = \frac{i}{2ab}\oint_C \frac{z^4 + 1}{z^2(z - a/b)(z - b/a)}dz$$

   and identify the contour C.
   (c) (5%) Apply residue theorem to find the value of I.

5. The wave equation describing the transverse vibration of a stretched membrane under tension T and having a uniform surface density ρ is

$$T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho\frac{\partial^2 u}{\partial t^2}$$

   (a) (15%)Find a separable solution to the above PDE for a membrane stretched on a frame of length a and width b (i.e., $u(0,y,t) = u(a,y,t) = 0$ and $u(x,0,t) = u(x,b,t) = 0$).
   (b) (5%) Show that the natural angular frequencies of such a membrane are given by

$$\omega^2 = \frac{\pi^2 T}{\rho}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right).$$

   where m and n are positive integers.