1. (20%) The following eigen value problem is given as

$$\nabla^2\Phi(x,y)=0$$

in the domain $0\le x\le\pi$ and $-h\le y\le 0$, where $h$ is positive constant, with the boundary conditions:

$$\frac{\partial\Phi}{\partial x}=0 \quad \text{for } x=0,\ -h\le y\le 0$$

$$x=\pi,\ -h\le y\le 0$$

$$\frac{\partial\Phi}{\partial y}=0 \quad \text{for } 0\le x\le\pi,\ y=-h$$

$$\frac{\partial\Phi}{\partial y}=\lambda\Phi \quad \text{for } 0\le x\le\pi,\ y=0.$$

(1) (14%) Find all the eigen values $\lambda_n$ and the corresponding eigen functions $\Phi_n$ in terms of $h$.

(2) (6%) Find $\lambda_n$ in the limits $h\to 0$, and also $h\to\infty$.

2. (20%) For a function $f(z)$ is given as:

$$f(z)=\frac{1}{z^2+z+1}\ln\left(\frac{z-i}{z+i}\right)$$

(1) (10%) Identify all singularities of $f(z)$, and specify their types.

(2) (10%) Using $z-i=r_1e^{i\theta_1}$, $z+i=r_2e^{i\theta_2}$. Is $f(z)$ single valued? Why and prove it.

3. (20%) Solve for $y(t)$ with $y(t)=u(t)+\int_0^t e^{-(t-\tau)}y(\tau)d\tau$ by using the Laplace Transform, where $u(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$.

(背面仍有題目,請繼續作答)

4. (a) (10%) Verify that $2x\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = 0$ has a solution $F(x,t) = f(x^{\alpha}t)$, where $f$ is arbitrary. Find the value of constant $\alpha$.
(b) (10%) Solve this equation by separation of variables. By using superposition, obtain again the result above for arbitrary $f$. How is the proper $\alpha$ obtained in this approach?

5. (1) (10%) The nonlinear system

$$u_t + uu_x + gy_x = 0$$

$$y_t + uy_x + yu_x = 0$$ has the solution $u = u_0$, $y = y_0$ (constants).

Linearize the system by introducing the small perturbations:

$u = u_0 + \varepsilon\omega(x,t)$, $y = y_0 + \varepsilon\eta(x,t)$ and neglect $\varepsilon^2$ and higher order terms. Find this linearized system.

(2) (10%) Following (1) above by eliminating $\eta$ from the system, find a second order linear differential equation for $\omega$ alone. Similarly, get an equation for $\eta$ alone and compare the two equations.