1. (a) 5% Does the matrix equation $A^2 = 0$ imply $A = 0$? Explain your answer.
   (b) 15% Find the most general 2×2 matrix whose square is zero.

2. Let $f(x) = (1+x)^m$, in which m may be negative and is not limited to integral values.
   (a) 10% Show $f(x)$ has the following binomial expansion :

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

   (b) 10% The total relativistic energy E of a particle with rest mass m and velocity v is known to

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

   where c is the speed of light. By using the result of part (a), show that for particle velocity $v \ll c$, the energy E can be approximated by

$$E = mc^2 + \frac{1}{2}mv^2$$

   where mc^2 is identified as the rest mass energy and $mv^2/2$ is the classical kinetic energy.

3. In this problem you are asked to evaluate the following definite integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}, \quad |a| < 1$$

   by applying residue theory.
   (a) 5% In terms of the variable $z = e^{i\theta}$ show that the above integral can be transformed into the following form

$$I = -i\frac{2}{a}\oint \frac{dz}{z^2 + (1/a)z + 1}$$

   (b) 15% Find the residue at the root of the denominator and show

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}$$

4. Considering the following differential equation

$$\frac{d^2y}{dx^2} + \omega^2 y = 0, \qquad \text{(A)}$$

   you are asked to find its series solution in the form of

$$y(x) = x^k \sum_{i=0}^{\infty} a_i x^i . \qquad \text{(B)}$$

   (a) 6% By substituting Eq. (B) into Eq. (A), show $k = 0$ or 1.
   (b) 7% When $k = 0$, find the values of a_i, $i \ge 2$ by assuming $a_0 \ne 0$ and $a_1 = 0$. In this case show that Eq. (B) gives $y(x) = a_0 \cos\omega x$.

（c） 7% When $k=1$, find the values of a_i, $i \geq 2$ by assuming $a_0 \neq 0$ and $a_1 = 0$. In this case show that Eq.（B） gives $y(x) = (a_0/\omega)\sin\omega x$.

5.（a） 10% Find a unit vector perpendicular to the surface

$$x^2 + y^2 + z^2 = 3$$

at the point $(1,1,1)$.

（b） 10% Derive the equation of the plane tangent to the surface at $(1,1,1)$.