1. (15%) Given that $x$ and $xe^x$ are solutions of the homogeneous equation corresponding to

$$x^2y'' - x(x+2)y' + (x+2)y = 2x^3,\quad x>0$$

Find the general solution by using the Method of Variation of Parameter.

2. (15%) Find the solution of the initial value problem

$$\frac{d^2x}{dt^2} + m^2x = \sum_{n=1}^{\infty} b_n \sin nt,\quad x(0)=0,\quad \frac{dx}{dt}(0)=0$$

where $m$ is a positive integer.

3. (15%) Given that Legendre polynomial satisfy the following two properties:

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x),\ P_0(x)=1,\ P_1(x)=x,\ n\ge 2$$

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & if\, n\ne m \\ \dfrac{2}{2n+1} & if\, n=m \end{cases}$$

please derive $R(n)$, where $\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = R(n)$.

4. (20%) By employing the Power Series Method for the equation of $-x^2y''+y''+xy'-y=0$, the solution can be expressed as the form of $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots,\ a_0 \ne 0$, please calculate the value of $\dfrac{a_4a_5}{a_3+a_4+a_5}$.

5. (15%) Prove: If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n\times n$ matrices, then $\det(\boldsymbol{AB})=\det(\boldsymbol{A})\det(\boldsymbol{B})$. (where "det" represents "determinant")

6. (a). (10%) Find an LU factorization of matrix $A = \begin{bmatrix} 2 & 2 & -2 & 5 & 2 \\ -2 & -1 & 2 & -3 & -1 \\ 4 & -1 & -5 & 6 & 6 \\ -4 & 4 & 6 & 2 & -1 \end{bmatrix}$

(b). (10%) Find the inverse matrix of $A = \begin{bmatrix} 3 & 1 & 2 & 6 & 5 \\ 1 & 2 & 0 & 4 & 3 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 3 & 2 \end{bmatrix}$