PART I.

1. Find the inverse P^{-1} of the matrix

$$P = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

if possible. (10%)

2. Let A be the matrix

$$\begin{pmatrix} 4 & -12 & 6 \\ 3 & -8 & 3 \\ 6 & -12 & 4 \end{pmatrix}$$

(a) Find the minimal polynomial of A. (10%)
(b) Find the nth power A^n of A, for every positive integer n. (10%)

3. Let A and B are nxn matrices, and p(x) is a polynomial such that $p(AB)=0$. Show that if $q(x)=xp(x)$, then $q(BA)=0$. (10%)

PART II.

1. Solve the following recurrence, where

$$T(1)=1$$
$$T(n)=2T(n/2)+n^2, \qquad \forall\ n\geqslant 2. \qquad (10\%)$$

2. Define the following terminologies:
(a) partial ordering, (3%)
(b) total ordering, (3%)
(c) well ordering. (2%)
Give an example that is partial ordering, but not total ordering. (4%)

3. How many integral solutions are there to

$$x_1+x_2+x_3+x_4=15$$

where $x_1\geqslant 1$, $x_2\geqslant 2$, $x_3\geqslant 3$, $x_4\geqslant 4$? (8%)

PART III.

1. Assume that 70% of an inventory of diodes comes from vendor 1 and the remaining 30% from vendor 2, and that 98% of the units from vendor 1 and 90% of those from vendor 2 give satisfactory performance. If we pick a diode form vendor at random, then
(a) What is the probability of selecting a unit that is made by vendor 1 and is also defective?
(b) What is the probability of selecting one that is defective, irrespective of vendor? (10%)

2. Let X_1 and X_2 be independent random variables each having the probability distribution

$$f(x)=\begin{cases} e^{-x}, & x>0 \\ 0, & \text{elsewhere} \end{cases}$$

Show that the random variables Y_1 and Y_2 are independent, where $Y_1=X_1+X_2$ and $Y_2=X_1/(X_1+X_2)$. (10%)

3. Assume that the dimensions, X and Y, of a rectangular metal plate may be considered to be independent continuous random variables with the following probability distributions

$$
\begin{array}{lll}
X: & g(x)=x-1, & 1<x\leqslant 2 \\
 & g(x)=-x+3, & 2<x<3 \\
 & g(x)=0, & \text{elsewhere} \\
Y: & h(y)=1/3, & 3<y<6 \\
 & h(y)=0, & \text{elsewhere}
\end{array}
$$

Find the probability distribution of the area of the plate A=XY. (10%)