Part I: Linear Algebra

1. Let

$$B = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Show that B is diagonalizable. (5%)

(b) Find B^{50}. (5%)

2. Find the least squares straight line, y=mx+b, fit to the four points (0,1), (2,0), (3,1), (3,2). (10%)

3. Consider the matrix (10%)

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 3 & 7 \end{bmatrix}$$

(a) Find elementary matrices E_1, E_2, and E_3 such that $E_1E_2E_3A = I$.

(b) Write A as a product of three elementary matrices.

4. If square matrices A and B are similar, prove that

(a) A and B have the same rank. (5%)

(b) A and B have the same trace. (5%)

5. Prove that if M is an m×n matrix with linearly independent columns, then M^tM is an n×n invertible matrix. (10%)

II. 離散數學

1. Consider the experiment of shooting at a target until there is a hit. The sample space is a countably infinite set S = {h, mh, mmh, ...} where h denotes a hit, mh denotes a miss followed by a hit, and so on. We assume the probability of occurrence of the sample that has k misses before a hit to be $2^{-(k+1)}$. Compute the amount of information in each of the following events.

   (a) There are no more than 5 misses followed by a hit. (4%)

   (b) There is a hit after an odd number of misses. (4%)

   (c) There is a hit after an even number of misses (including no misses). (4%)

2. Show that $2^n > n^3$ for $n \geq 10$. (10%)

3. If there is an unlimited number (or at least 25 of each color) of red, green, white and black balls, in how many ways can you select 25 of these balls so that you have an even number of white balls and at least six black balls ? (10%)

4. Let W = {a,b,c,d}. Define two binary operations + and * as follows.

+	a	b	c	d
a	d	c	a	b
b	c	d	b	a
c	a	b	c	d
d	b	a	d	c

*	a	b	c	d
a	d	d	c	c
b	d	d	c	c
c	c	c	c	c
d	c	c	c	c

From these tables, we see that the associative laws of both + and * hold.

(a) Is (W, +, *) an integral domain ? Why ? (4%)

(b) What kind of algebraic structure does (W, +, *) belong to ? Why ? (6%)

5. Define the following terms in the Graph Theory.

   (a) Hamiltonian Path. (4%)

   (b) Isomorphic Graphs. (4%)