Part I : Linear Algebra (50%)

1. Compute the ranks of the following matrices. (10%)

$$(1)\ A = \begin{bmatrix} 3 & 1 & 2 & 5 \\ 1 & 2 & -1 & 2 \\ 4 & 3 & 1 & 7 \end{bmatrix} \qquad (2)\ B = \begin{bmatrix} 3 & 1 & 2 & 5 \\ 1 & 2 & -1 & 2 \\ 4 & 3 & 1 & 1 \end{bmatrix}.$$

2. Show that the vectors (1, 1) and (−1, 2) form a basis of R^2. (5%)

   Find the coordinates of the vector (a, b) in R^2 with respect to the vectors (1, 1) and (−1, 2). (5%)

3. A cryptosystem is one in which a meaningful message block M, called the plaintext, is enciphered (or transformed) into a meaningless message block, called the ciphertext. This transformation is usually specified by a key in shch a way that only the authorized users who know the key can decipher (recover) the ciphertext.

   Assume that you are an attacker. Knowing that the transformation of a given cryptosystem is a 3*3 matrix transformation and also knowing the following pairs of plaintex–ciphertext:

$$(M_1, C_1) = (2, 1, 2), (3, 15, 8);$$
$$(M_2, C_2) = (0, 6, -2), (-6, 14, -2);$$
$$(M_3, C_3) = (0, 3, -1), (-3, 7, -1);$$
$$(M_4, C_4) = (4, 1, 1), (7, 21, 13).$$

   (a) Find the enciphering and deciphering keys of the cryptosystem. (10%)
   (b) Compute the plaintext of the ciphertext C = (0, 8, 4). (5%).
   (c) What is the requirement(s) for a matrix to be a transformation in a cryptosystem. (5%)

4. Consider the vector space of all functions of a variable t. Show that the pair of functions e^t, e^{2t} are linear independent. (10%)

Part II　Discrete Mathematics

1. (15%)　Solve the following recurrence relation for the Fibonacci sequence of numbers :

$F_n = F_{n-1} + F_{n-2}$, given that $F_1 = F_2 = 1$.

2.(10%)　For the finite state machine shown below, please find all equivalent states and obtain an equivalent finite state machine with the smallest number of states.

Present state	Next state: Present input 0	Next state: Present input 1	Output
A	F	B	0
B	D	C	0
C	G	B	0
D	E	A	1
E	D	A	0
F	A	G	1
G	C	H	1
H	A	H	1

3. (15%)　Show that a group in which $x \cdot x = i$ for each $x \in G$ is communicative.

4. (10%)　Sow that there exists a constant c such that for $n \geq c$, $(1.1)^n \geq n^{100}$.