Part I. Linear Algebra

I. Notations and definitions

In the following problem set, the symbols $\mathbb{R}$ and $\mathbf{C}$ will be reserved for the set of all real numbers and the set of all complex numbers, respectively. For a field K and for any positive integer n, $\mathrm{Mat}_n(K)$ denotes the set of all n by n matrices over K. For a matrix $A \in \mathrm{Mat}_n(K)$, A^t is the transpose of A.

II. Problems

(1) Let $n \geq 1$. Show that if $A \in \mathrm{Mat}_n(\mathbb{R})$ and $A^tA = 0$, then $A = 0$. 6%

(2) Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, $n > m \geq 1$.
  (i) Show that A need not have a right inverse. 6%
  (ii) Suppose that A does have a right inverse. Show that A has more than one right inverses. 10%

(3) Let A be the real matrix 10%

$$A = \begin{pmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{pmatrix}.$$

Find $P \in \mathrm{Mat}_3(\mathbb{R})$ such that $P^{-1}AP$ is a diagonal matrix.

(4) The vectors $v_1 = (1,1,1)$, $v_2 = (1,1,-1)$ and $v_3 = (1,-1,-1)$ form a basis of the vector space $\mathbf{C}^3$. Let $\{u_1, u_2, u_3\}$ be a dual basis of $\{v_1, v_2, v_3\}$ and let $v = (0,1,0) \in \mathbf{C}^3$. Find the inner products $\langle v, u_1\rangle$, $\langle v, u_2\rangle$ and $\langle v, u_3\rangle$. 8%

(5) Let $y_0, y_1, y_2, \ldots$ be the sequence of the Fibonacci numbers where $y_0 = 0$, $y_1 = 1$ and $y_{n+1} = y_n + y_{n-1}$ for all $n \geq 2$. Let $z_n = y_{n-1}$ for $n \geq 1$. Then the Fibonacci sequence can be written as a first order recurrences system 10%

$$y_{n+1} = y_n + z_n,$$
$$z_{n+1} = y_n$$

with initial conditions $y_1 = 1$ and $z_1 = 0$. By setting $\mathbf{y}_n = (y_n, z_n)^t$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, one obtain

$$\mathbf{y}_{n+1} = A\mathbf{y}_n.$$

Now, diagonalize A and obtain a formula for the $(n+1)$-th Fibonacci number y_n.

Part II: Discrete Mathematics (50%)

1. (10%) Let $\{0,1\}$ be the input and output alphabet. Draw the state diagram of a finite state machine that reverses (from 0 to 1 or from 0 to 1) the symbols appearing in the 4th, in the 8th, in the 12th, ... , positions of an input string. For example, if ω is the output function of the finite state machine, then $\omega(0000) = (0001)$, $\omega(000111) = (000011)$ and $\omega(000000111) = (000100101)$. When drawing the state diagram, assume that s_0 is the starting state.

2. (10%) At a high school science fair, 34 students received awards for scientific projects. Fourteen awards were given for projects in biology, 13 in chemistry and 21 in physics. If three students received awards in all three subject areas, how many received awards for exactly (a) one subject area? (b) two subject areas?

3. (10%) Construct a formal proof for the following theorem:

   If $(p \vee r)$ and $(q \vee \sim r)$, then $(p \vee q)$.

4. (10%) In a certain population model, the probability that a couple will have n children satisfies the recurrence relation $p_n = 0.6p_{n-1},\ n \geq 1$. Knowing that $\Sigma_{i=0}^{\infty} p_i = 1$, what is the probability of a couple having no children?

5. (10 %) Construct an optimal prefix code following Huffman's recursive procedure for the symbols a, o, q, u, y, z that occur with frequencies 20,28,4,17,12,7, respectively.