Part I. Linear Algebra

1. Explain the following terms. (10%)
   (a) Rank equation
   (b) Singular matrix
   (c) Row echelon form
   (d) Homogeneous system

2. Let $\mathbf{u}=[-1,2]$ and $\mathbf{v}=[3,-5]$ be in R^2 (Euclidean 2-space), and let $T: R^2 \to R^3$ be a linear transformation such that $T(\mathbf{u})=[-2,1,0]$ and $T(\mathbf{v})=[5,-7,1]$. Find the standard matrix representation A of T and compute $T([-4,3])$. (10%)

3. Find the least-squares solution of the given overdetermined system $A\mathbf{x}=\mathbf{b}$ by converting it to a consistent system and then solving. (10%)

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

4. Find the eigenvalues λ_i and the corresponding eigenvectors $\mathbf{v}_i$ of the given matrix A; and also find an invertible matrix C and a diagonal matrix D such that $D=C^{-1}AC$. (10%)

$$A=\begin{bmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{bmatrix}$$

5. Find an orthonormal basis for a subspace in R^4 being spanned by $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, i.e. $W=\mathrm{sp}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$, if $\mathbf{v}_1=[1,1,1,1]$, $\mathbf{v}_2=[-1,1,-1,1]$, and $\mathbf{v}_3=[1,-1,-1,1]$. Then find the projection of $\mathbf{b}=[1,2,3,4]$ on W. (10%)

(背面仍有題目,請繼續作答)

Part II. Discrete Mathematics (50%)

1. [10%] Let A and B be two countable **infinite** sets in which $A \neq B$. Define $A \oplus B = \{3x \mid x \in A \text{ and } 2x \in B, \forall x\}$. Let $f: A \rightarrow B$, $g: A - B \rightarrow A \cup B$, and $h: A \oplus B \rightarrow B$ be three functions.
   a. Please show an example of A, B, *f*, *g*, and *h* such that *f*, *g*, and *h* are one-to-one and onto.
   b. Please show an example of A, B, *f*, *g*, and *h* such that *f*, *g*, and *h* are one-to-one **but not** onto.
2. [20%] You are given a boolean function, C_test(G), that performs the connectivity test of any input graph G. If G is connected (namely, there exists a path for any two nodes in G), then C_test(G) returns true; otherwise, C_test(G) returns false.

   Define the fault-tolerant degree, *ft*(G), of a graph G to be the number of link(s) that can be removed without effecting the connectivity of the resulting graph. In other word, if *ft*(G) = *k*, then the removal of *any k* links from G does not affect the connectivity of the resulting graph. For example, if G is a ring, then *ft*(G) = 1. Namely, if *any one* link is removed from G, the new graph is still connected.
   a. Please draw a graph with 6 nodes such that *ft* (G) = 2.
   b. For a graph G with *n* nodes, what is the minimum number of links to make *ft*(G) = *k* ?
   c. Define FT_test(G,*k*) to be the function that returns true if *ft* (G) $\geq k$. Namely if *ft* (G) $< k$, then FT_test(F, *k*) returns false. Let the complexity of C_test(G) be $O(n^3)$ for a graph G with *n* nodes and α links. Please derive a recurrence relation for FT_test(G, *k*) in terms of FT_test(G,*k-1*), FT_test(G,*k-2*), ..., FT_test(G,*1*) .
   d. (Continued from c.) Solve the recurrence relation. What is the complexity of FT_test(G,*k*)?
3. [20%] Assume the statistics show that if today is SUNNY, the probability of being SUNNY, CLOUDY, and RAINY tomorrow is 0.3, 0.3, and 0.4 respectively. If today is CLOUDY, then the probability of being SUNNY, CLOUDY, and RAINY tomorrow is 0.2, 0.2, and 0.6 respectively. Furthermore, if today is RAINY, then the probability of being SUNNY, CLOUDY, and RAINY tomorrow is 0.2, 0.3, and 0.5 respectively.
   a. If today is SUNNY, what is the probability of having *exactly* three consecutive SUNNY days (including today)?
   b. You are instructed to carry an umbrella on a trip if the probability of being RAINY, for *any day* on the trip, is greater than or equal to 0.5. Given that today is RAINY and you plan to have a trip for 3 days (starting from tomorrow), will you carry an umbrella with you on the trip?