Part I. Linear Algebra (50%)

1. (a) Find all numbers r such that the matrix $\begin{bmatrix} 2 & 4 & 2 \\ 1 & r & 3 \\ 1 & 1 & 2 \end{bmatrix}$ is invertible. (5%)

   (b) Find the determinant of the matrix $\begin{bmatrix} 1 & 2 & 0 & -1 & 2 & 4 \\ 6 & 2 & 8 & 1 & -1 & 1 \\ 4 & 2 & 1 & 2 & 2 & -5 \\ 4 & 5 & 4 & 5 & 1 & 2 \\ 1 & 2 & 0 & -1 & 2 & 4 \\ 1 & 0 & 1 & 8 & 1 & 5 \end{bmatrix}$. (5%)

2. (a) Find the rank of matrix $\begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}$. (7%)

   (b) Find the nullity of matrix $\begin{bmatrix} 0 & -9 & -9 & 2 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}$. (8%)

3. (a) Find an orthogonal basis for the subspace $\mathrm{sp}(1, \sqrt{x}, x)$ of the vector space $C_{0,1}$ of continuous functions with domain $0 \le x \le 1$, where inner product is defined by $< f, g >= \int_0^1 f(x)g(x)dx$. (10%)

   (b) Solve the linear differential system $\begin{aligned} x_1' &= x_1 - x_2 - x_3 \\ x_2' &= -x_1 + x_2 - x_3 \\ x_3' &= -x_1 - x_2 + x_3 \end{aligned}$. (15%)

(背面仍有題目,請繼續作答)

Part II. Discrete Mathematics (50%)

Note: (Write your answers on the answer sheet but not this question sheet.)

1. [5%] Given three arbitrary sets A, B, and C, please indicate whether each of the following statements is true or false:

____ a. $A \oplus A = \varnothing$
____ b. If $A = B - C$ then $B = A \cup C$
____ c. $(A - B) \cap B = B$
____ d. $(A \cup B) - C = (A - C) \cup (B - C)$
____ e. $(A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cap C$

2. [10%] Let R be a binary relation. Let D_1 and D_2 be the domains of the two components of R. Define $D = D_1 \cup D_2$. Please indicate whether each of the following statements is true or false.

____ a. R is reflexive if and only if $\forall x \in D\ (x R x)$.
____ b. R is irreflexive if and only if $\neg (\forall x \in D\ (x R x))$.
____ c. Any relation must be either reflexive or irreflexive.
____ d. R is symmetric if and only if $\forall x, y \in D\ (x R x \Rightarrow y R y)$.
____ e. R is transitive if and only if $\forall x, y, z \in D\ (x R y \wedge y R z \Rightarrow x R z)$.

3. [15%] Let $N = \{1, 2, ..., n\}$. Let $P = (X_1, X_2, ..., X_n)$ be a permutation of members in N if and only if

(1) $X_i \neq X_j$ if $i \neq j$ and (2) each $X_i \in N$.

We call $j_1, j_2, ..., j_k$ a cycle of P, if and only if

(a) $j_i \neq j_m$ for every $i \neq m$ unless $i = 1$ and $m = k$
(b) $j_1 = j_k$
(c) $j_{i+1} = X_{j_i}$ $1 \leq i < k$.

For example, consider the permutation P = (2,3,6,5,4,1,7). The cycle 1,2,3,6,1 satisfies conditions (a), (b), and (c). The permutation (2,3,6,5,4,1,7) has the three cycles:

1,2,3,6,1,
4,5,4, and
7,7.

Let R be a binary relation. For any permutation P of N, R is defined as R = {(a, b) | a and b are in the same cycle of P}. Please answer yes (Y) or no (N) to each of the following questions. You will NOT get full grade UNLESS you justify (e.g. give an explanation to) each question.

____ a. Is R reflexive?
____ b. Is R symmetric?
____ c. Is R transitive?
____ d. Does the permutation (2, 4, 5, 7, 1, 8, 3, 6) have at least four cycles?
____ e. Is the cycle 4, 5, 4 in the permutation (2 , 4, 5, 3, 1)?

4. [20%] Solve the following recurrence relations.

a. $f(n) = k\, f(n/k) + n^* \log_k n$, where n is a power of k and $f(1) = 1$.
b. Please compute g(10). $g(n) = 3\, g(n-1) + 2g(n-2) + g(n-3) + 4$, $g(0) = 0$, $g(1) = 2$, and $g(2) = 3$.