Part I. Linear Algebra (50%)

1. Determine all values of the b_i that make the following linear system consistent. (10%)

$$x_1 + x_2 - x_3 = b_1$$
$$2x_2 + x_3 = b_2$$
$$x_2 - x_3 = b_3$$

2. (a) Explain and determine whether the matrices A and B are diagonalizable. (5%)

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 & 5 & 1 \\ 2 & 0 & 2 & 6 \\ 5 & 2 & 7 & -1 \\ 1 & 6 & -1 & 3 \end{bmatrix}$$

(b) Find the unique solution (assuming that it exists) of the system of equations expressed by the partitioned matrix. (10%)

$$\left[\begin{array}{cccc|c} a_1 & b_1 & c_1 & d_1 & 3b_1 \\ a_2 & b_2 & c_2 & d_2 & 3b_2 \\ a_3 & b_3 & c_3 & d_3 & 3b_3 \\ a_4 & b_4 & c_4 & d_4 & 3b_4 \end{array}\right]$$

3. (a) Supply a third column vector so that the following matrix is orthogonal. (5%)

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & \\ 1/\sqrt{3} & 0 & \\ 1/\sqrt{3} & -1/\sqrt{2} & \end{bmatrix}$$

(b) Find the least-squares fit of the data points (-3, 8), (-1, 5), (1, 3), and (3, 0) by a straight line, i.e. by a linear function $y = r_0 + r_1 x$. (10%)

4. Consider the vector space P_2 of polynomials of degree at most 2, and let B' be the ordered basis $(1, x, x^2)$ for P_2. Let $T: P_2 \to P_2$ be the linear transformation such that $T(1) = 3 + 2x + x^2$, $T(x) = 2$, $T(x^2) = 2x^2$. Find $T^4(x+2)$. (10%)

(背面仍有題目,請繼續作答)

Part II. Discrete Mathematics (50%)

1. (15%) Let k is an integer and $k > 4$. If $G = [V, E]$ is a connected planar graph with vertices $|V| = v$, edges $|E| = e$, and each cycle of length at least k,
   (a). What is the minimal cycle length in the graph $K_{3,3}$?
   (b). Prove that $e\,(k - 2) \leq (v - 2)\,k$
   (c). Is $K_{3,3}$ planar ? Explain why.

2. (15%) A company purchased 10,000 resistors, in which 3,000 are from agent A, 5,000 from agent B and 2,000 from agent C, respectively. Assume that 2%, 3% and 4 % of the resistors from agents A, B and C are defected, respectively. For a resistor selected randomly from the 10,000 resistors,
   (a). What is the probability that the selected resistor is defected?

   (b). If it is known that the selected resistor is not from agent A, what is the probability that the selected resistor is defected?

   (c). If the selected resistor is known to be defected, what is the most possible agent (call it agent X) the selected resistor is from? Give the probability the selected resistor is from agent X.

3. (10%) Use generating functions to determine in how many ways can we select seven nonconsecutive integers from {1, 2, 3, …., 50} ?

4. (10%) Let $A = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$, and define $\Re$ on A by $(x_1, y_1)\ \Re\ (x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$.
   (a). Prove or disprove that $\Re$ is an equivalence relation on A.
   (b). How many cells are there in the partition of A induced by $\Re$?