Part I. Linear Algebra (50%)

1. (10 %) A matrix A is said to be skew symmetric if $A^T = -A$. Let A be an $n \times n$ skew symmetric matrix.

   (a) If $A = B+C$ where B is symmetric and C is skew symmetric, find B and C. (5%)

   (b) If n is odd, show that A must be singular. (5%)

2. (15%) Let $C = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ and $D = [\mathbf{b}_1, \mathbf{b}_2]$, where $\mathbf{u}_1 = (1, 0, -1)^T$, $\mathbf{u}_2 = (1, 2, 1)^T$, $\mathbf{u}_3 = (-1, 1, 1)^T$ and $\mathbf{b}_1 = (1, -1)^T$, $\mathbf{b}_2 = (2, -1)^T$. For each of the following linear transformations L from R^3 to R^2, find the matrix representing L with respect to the ordered bases C and D.

   (a) $L(\mathbf{x}) = (x_3, x_1)^T$ (5%)

   (b) $L(\mathbf{x}) = (x_1 + x_2, x_1 - x_3)^T$ (5%)

   (c) $L(\mathbf{x}) = (2x_2, -x_1)^T$ (5%)

3. (10%) Let A be a 7×5 matrix with rank equal to 4 and let $\mathbf{b}$ be a vector in R^8. The four fundamental subspaces associated with A are $R(A), R(A^T), N(A)$, and $N(A^T)$.

   (a) What is the dimension of $N(A^T)$ (2%) and which of the other fundamental subspaces is the orthogonal complement of $N(A^T)$ (3%)?

   (b) What is the dimension of $N(A^TA)$ (2%)? How many solutions will the least squares system $A\mathbf{x} = \mathbf{b}$ have? (3%).

4. (15%) Solve the initial value problem $\mathbf{Y}' = A\mathbf{Y}, \mathbf{Y}(0) = \mathbf{Y}_0$, where

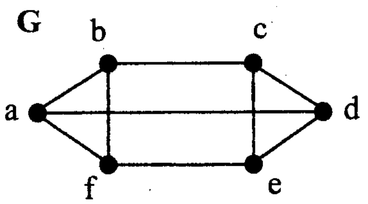
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \quad \mathbf{Y}_0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

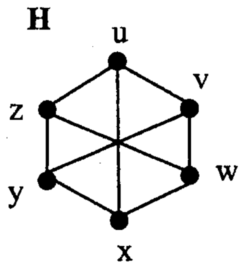
Part II. Discrete Mathematics 2007 (50%)

1. [10%] In how many ways can be the integers 1, 2, 3,…, n be arranged in a line so that none of the patterns 12, 23, 34,…, $(n-1)n$ occurs?

2. [15%] Let S be a set containing n distinct objects. Verify that $e^x/(1 - x)^k$ is the exponential generating function for the number of ways to choose m of the objects in S, for $0 \le m \le n$, and distribute these objects among k distinct containers, with the order of the objects in any container relevant for the distribution.

3. [15%] Decide whether the graphs **G** and **H** are isomorphic. Prove that your answer is correct.





4. Let $f,g : Z^+ \to R$, where $f(n) = 4^{\lg n} + n + 3n^{1/\lg n}$, $g(n) = \begin{cases} 1, & \text{for } n \text{ odd} \\ n^5, & \text{for } n \text{ even} \end{cases}$,

   (1) [5%] Choose correct statements in the following:
   (a) $f(n) = O(n)$, (b) $f(n) = O(n^2)$, (c) $f(n) = O(n^3)$,
   (d) $f(n) = \Omega(n)$, (e) $f(n) = \Omega(n^2)$, (f) $f(n) = \Omega(n^3)$,
   (g) $f(n) = \Theta(n)$, (h) $f(n) = \Theta(n^2)$, (i) $f(n) = \Theta(n^3)$.

   (2) [5%] Verify that $f \notin O(g)$ and $g \notin O(f)$.