Part I. Linear Algebra (50%)

(Explain or show your work for full credit)

1. (30%) Basics

(a) A 4x4 matrix A has eigenvalues $\lambda_1>\lambda_2>\lambda_3>\lambda_4>0$. Let D=diag([λ_1 λ_2 λ_3 λ_4]) and E be an orthogonal matrix whose columns are ordered eigenvectors belonging to λ_1, λ_2, λ_3, λ_4, respectively.

(i) Which of the following matrices are singular? (0-5 choices) (5%)

(ii) Which of the following matrices are similar to A? (0-5 choices) (5%)

(1) $A-\lambda_1 I$ (2) e^A (3) A^TA (4) $ED^{-1}E^T$ (5) $\begin{bmatrix} \lambda_4 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$.

(b) Determine which sets of vectors are linearly independent. (0-4 choices) (10%)

(1) The vectors $x+1,\ x^2-x+1,\ x^2+3x-2,\ 2x^2+8x+1$ in P_3.

(2) The vectors $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 7 & 3 \end{bmatrix}$ in R^{2x2}.

(3) The vectors $e^x+e^{-x},\ e^x-e^{-x},\ e^{2x}$ in $C[0,1]$.

(4) The eigenvectors of the matrix $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

[Note] P_3: the vector space of polynomials of degree less than 3.
C[0,1]: the vector space of all real-valued functions that are continuous on the interval [0,1].

(c) $\mathbf{v}_1=[1\ 2\ 4\ 8\ 16\ 32]^T$ and $\mathbf{v}_2=[1\ -2\ 4\ -8\ 16\ -32]^T$. What is the rank of $\mathbf{v}_1\mathbf{v}_1^T+\mathbf{v}_2\mathbf{v}_2^T$? (5%)

(d) Let $V=Span([0\ 1\ 1]^T,[0\ 0\ 1]^T)$, $U_1=\{[1\ y\ z]^T \mid y,z\in R\}$, $U_2=\{[1\ 0\ 0]^T\}$, $U_3=V^{\perp}$, and $U_4=N([0\ 1\ 1])$. Which U_i satisfies the direct sum $U_i\oplus V=R^3$? (0-4 choices) (5%)

[Note] $V^{\perp}$: the orthogonal complement of V.
N(U): the nullspace of U.

（背面仍有題目,請繼續作答）

2. (20%) The four vertices (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4) of a polygon form a 2x4 matrix $X=\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$.

(a) The transformation $L(X) = 2X+\begin{bmatrix} 7 \\ 7 \end{bmatrix}\mathbf{1}^T$ represents a dilation and a translation of a polygon with vertex matrix X, where $\mathbf{1}=[1\ 1\ 1\ 1]^T$. Explain why L is not a linear transformation from R^{2x4} to R^{2x4}. (5%)

(b) To perform the dilation and the translation in (a) as a linear transformation, one can attach a third row to the vertex matrix X as $Y=\begin{bmatrix} X \\ \mathbf{1}^T \end{bmatrix}$ and change the transformation L to be $L_2(Y) = AY$. Simply ignoring the third row of Y gives the desired vertex matrix. What is the transformation matrix A? (5%)

(c) Consider the approximation of a polygon $X=\begin{bmatrix} 0 & 1 & 1 & -1 \\ -1/2 & 0 & 1 & 0 \end{bmatrix}$ by a circle $(x-a)^2+(y-b)^2=r^2$ and by a quadratic polynomial $y=c_0+c_1x+c_2x^2$. Which approximation gives a better least squares fit to the polygon? How large is the residual? (10%)

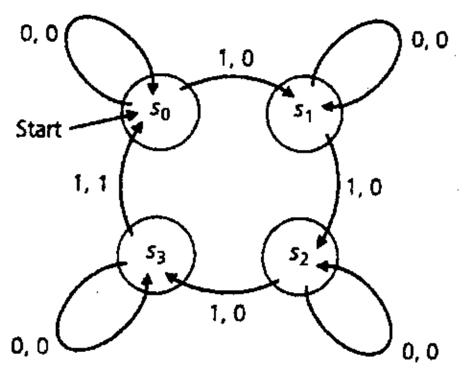
Part II. Discrete Mathematics 2006 (50%)

1. [10%] At a certain company, the manager has a secretary and three other administrative assistants. If seven accounts must be processed, in how many ways can the manager assign the accounts so that each assistant works on at least one account and the secretary must handle the most expensive account?

2. [10%] Determine the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 19$ where $-5 \le x_i \le 10$ for all $1 \le i \le 4$.

3. [10%] Show that $(1 - 4x)^{-1/2}$ generate the sequence $\binom{2n}{n}$, $n \in \mathbf{N}$.

4. Let M be the finite state machine in the following figure.



(a) [5%] Find the state table for this machine.

(b) [5%] Explain what this machine does.

(c) [10%] How many distinct input strings x are there such that $\|x\| = 8$ and $\nu(s_0, x) = s_0$? How many distinct input strings x are there such that $\|x\| = 12$ and $\nu(s_0, x) = s_0$?