1. Find a general solution of the equation

$$x^2y'' + (1-K-K^{-1})xy' + y = 0$$

(10%)

2. Using $\mathcal{L}(t\cos\omega t)$ show that $\mathcal{L}^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right) = \frac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$

(8%)

3. Solve the initial value problems involving system of differential eqs.

$$\begin{cases} y_1'' + y_2 = -5\cos 2t \\ y_2'' + y_1 = 5\cos 2t \end{cases}$$

(by the Laplace transform)

and $y_1(0)=1$, $y_1'(0)=1$, $y_2(0)=-1$, $y_2'(0)=1$ (10%)

4. The vertical vibration of the mechanical system is governed by the system of differential equations.

$$\begin{cases} \ddot{y}_1 = -ky_1 + k(y_2-y_1) \\ \ddot{y}_2 = -k(y_2-y_1) - ky_2 \end{cases}$$

(a) Find the eigenvalues and eigenvectors (5%)

(b) Find the solution of the eqs that satisfy the initial conditions $y_1(0)=1$, $y_2(0)=1$, $\dot{y}_1(0)=\sqrt{3k}$, $\dot{y}_2(0)=-\sqrt{3k}$ (7%)

(c) What does it means mechanically that the eigenvalues are negative (4%)

5. Using Stokes' theorem, evaluate $\int_C v_t \, ds$

i.e. $$\int_S (\mathrm{curl}\,\vec{v})_n \, dA = \int_C v_t \, ds$$

where $(\mathrm{curl}\,\vec{v})_n = (\mathrm{curl}\,\vec{v})\cdot\vec{n}$

given: $\vec{V} = 2y\vec{i} + z\vec{j} + 3y\vec{k}$ (12%)

C: the intersection of $x^2+y^2+z^2=6z$ and $z=x+3$, oriented in the clockwise sense as viewed from the origin.

6. Using the Fourier integral representation, show that

$$\int_0^\infty \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2}\, d\omega = \begin{cases} 0 & \text{when } x<0 \\ \frac{\pi}{2} & \text{when } x=0 \\ \pi e^{-x} & \text{when } x>0 \end{cases} \quad (8\%)$$

7. The small free vertical vibrations of a uniform cantilever beam are govened by

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^2 u}{\partial x^4} = 0 \qquad \text{where } c^2 = \frac{EI}{\rho A},$$

E: Yong's modulus
I: inertia moment
ρ: density
A: cross section

Find solution that satisfies the boundary conditions

$u(0,t)=0$, $u(\ell,t)=0$

$u_{xx}(0,t)=0$, $u_{xx}(\ell,t)=0$ where ℓ: beam length.

and the initial condition

$u(x,0)=f(x)=x(\ell-x)$

$\dot{u}(x,0)=0$ (14%)

8. Integrate $1/(z^4-1)$ in the counterclockwise sense around the circle $|z+1|=1$ (8%)

9. Show that $y'=(y/x)+1$ can not be solved for y as a power series in x. (8%)

but, this equation can be solved for y as a power series in powers of $x-1$. (6%)

(hint: Introduce $t=x-1$ as a new independent variable and solve the resulting equation for y as a power series in t)