編號： 40

國立成功大學 108 學年度碩士班招生考試試題

系　所：物理學系

考試科目：物理數學

考試日期：0224，節次：1

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. Let A be a Hermitian matrix, U a unitary matrix, and $\mathbb{R}$ the set of real numbers, show that

(1) $V = \mathrm{e}^{\mathrm{i}A}$ is a unitary matrix; (5%)

(2) if $U\vec{v} = \lambda\vec{v}$ for some $\vec{v}^\dagger\vec{v} \neq 0$, then the corresponding eigenvalue $\lambda = \mathrm{e}^{\mathrm{i}\phi}$ for some $\phi \in \mathbb{R}$; (7%)

(3) if A is both unitary and Hermitian, then it can only have eigenvalues from the set $\{\pm 1\}$. (6%)

2. Let $\Psi(\vec{r},t) = \Psi_1(\vec{r},t)$ and $\Psi(\vec{r},t) = \Psi_2(\vec{r},t)$ be two *distinct* solutions to the linear differential equation

$$\alpha\frac{\partial\Psi(\vec{r},t)}{\partial t} = \beta\nabla^2\Psi(\vec{r},t) + \gamma\Psi(\vec{r},t),$$

where $\vec{r}$ is the position vector, t is the time variable, ∇^2 is the Laplacian operator, $\alpha, \beta, \gamma \in \mathbb{R}$ are constants independent of $\vec{r}$ and t.

(1) Show that arbitrary superpositions of these solutions, i.e., arbitrary linear combinations of $\Psi_1(\vec{r},t)$ and $\Psi_2(\vec{r},t)$ are also solutions to the differential equation; (9%)

(2) Show that the above partial differential equation (PDE) can be solved by using the method of *separation of variables*. Specifically, show that this method leads to four ordinary differential equations (ODE) corresponding to each variable. Provide a general solution to each of these ODEs. (13%)

3. A certain central force field is described, for $r > 0$, by the potential function $\Phi(\vec{r}) = \frac{k}{r}$ where $\vec{r} = (r, \theta, \phi)$ is the position vector with respect to some position in space, k is a constant.

(1) Determine the force field $\vec{E}(\vec{r}) = -\nabla\Phi(r)$ for $r > 0$ by computing the gradient of the potential $\Phi(r)$. (6%)

(2) Determine the divergence of $\vec{E}(\vec{r})$ for $r > 0$. (5%)

(3) Use Stokes' theorem or otherwise to show that for any close path c, $\oint_c \vec{E}(\vec{r}) \cdot d\vec{\ell} = 0$. (9%)

4. The periodic function $f(x)$ equals 1 whenever $x - n$ lies within the interval $[-\frac{1}{4}, \frac{1}{4}]$ for some integer $n \in \mathbb{Z}$, but vanishes otherwise.

(1) Plot $f(x)$ for $x \in [-1, 1]$ (label the axes of your plot clearly and include appropriate tick marks to indicate the limits of $f(x)$ and the values of $x \in [-1, 1]$ at which $f(x)$ is discontinuous). Use the plot to determine the period — *smallest* non-negative number λ such that $f(x + \lambda) = f(x)$ for all x? (11%)

(2) What are the coefficients a_0, a_k and b_k for $k = 1, 2, \ldots$ if we are to write $f(x)$ in its Fourier series (16%) :

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2k\pi x}{\lambda} + b_k \sin\frac{2k\pi x}{\lambda}\right)? \qquad (1)$$

5. Let $z = x + \mathrm{i}y$ (with $x, y \in \mathbb{R}$) be a complex variable and $f(z) = \mathrm{e}^z/(z-1)$.

(1) What is the Taylor series of e^{z-1} at $z = 1$? Use this expansion to obtain the Laurent series of $f(z)$ about $z = 1$ and argue that $f(z)$ is analytic everywhere on the complex plane except at $z = 1$. (4%)

(2) What is the residue of of $f(z)$ at $z = 1$? Use this to evaluate $\oint_c f(z)\mathrm{d}z$ where c is the close square path that traverses sequentially through the points $z = -\mathrm{i}$, $z = \mathrm{i}$, $z = 2 + \mathrm{i}$, $z = 2 - \mathrm{i}$, and $z = -\mathrm{i}$. (7%)

(3) Consider now a different path c' defined by traversing sequentially through the points $z = -\mathrm{i}$, $z = 0$, $z = 2 - \mathrm{i}$, and $z = -\mathrm{i}$. What is the value of $\oint_{c'} f(z)\mathrm{d}z$? (2%)