國立成功大學 109 學年度碩士班招生考試試題

系　所：物理學系

考試科目：物理數學

考試日期：0211，節次：1

※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1: Let $\mathbf{A}$ and $\mathbf{B}$ be two non-zero $d \times d$ matrices (with d being a positive integer) that satisfy $(\mathbf{A}\,\mathbf{B})^{\dagger} + \mathbf{B}^{-1}\mathbf{A} = \mathbf{0}$.

(1) Prove that if $\mathbf{B}$ is unitary and if $\mathbf{A}$ is real, then $\mathbf{A}$ is antisymmetric. (5%)

(2) *Without* assuming that $\mathbf{B}$ is unitary, prove that if both $\mathbf{A}$ and $\mathbf{B}$ are real and if d is an odd number, then $\mathbf{A}$ must be singular. (13%)

2: Let $\vec{n} = (n_x, n_y, n_z) \in \mathbb{R}^3$ and $\mathbf{C} = \begin{pmatrix} 1+n_z & n_x - i n_y \\ n_x + i n_y & 1 - nz \end{pmatrix}$. Compute the determinant $|\mathbf{C}|$ and the trace of $\mathbf{C}$ and use them to show that $\mathbf{C}$ has *only* non-negative eigenvalues if and only if $|\vec{n}|^2 \leq 1$. (11%)

3: For the scalar function $\phi(x,y,z) = (x^2+y^2+z^2)\mathrm{e}^{-(x^2+y^2+z^2)}$,

(1) rewrite it in the spherical polar coordinates and compute its gradient; (10%)

(2) determine the coordinates at which the gradient vanishes; (6%)

(3) determine the directional derivative along $\hat{n} = \frac{3}{5}\hat{e}_x + \frac{4}{5}\hat{e}_y$ at the Cartesian coordinates of $x = x_0, y = 0, z = 0$ ($x_0 > 0$) where $\hat{e}_x, \hat{e}_y$ are unit vectors pointing along, respectively, the x-axis and the y-axis. (9 %)

4. In solving the Laplace equation in plane polar coordinates using the method of separation of variables, one arrives at the linear, ordinary differential equation (ODE) of $R(r)$: $r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} - n^2R = 0$ with n being a positive integer.

(1) Solve the ODE and find its *most* general solution $R(r)$ (without imposing any boundary conditions). (7%)

(2) How does the requirement of $R(r)$ remains finite (i.e., $|R(r)| < \infty$) when (i) $r \to 0$ (ii) $r \to \infty$ *each* affect the solution $R(r)$ allowed? (2%)

(3) Find a particular solution $R(r)$ satisfying the boundary conditions that $R(1) = 0$ and $R(\mathrm{e}) = 1$. (6%)

5: For the family of functions $f_n(x) = x^n$ parametrized by a non-negative integer $n = 0, 1, 2, \ldots$,

(1) determine a general expression for the Laplace transform of $f_n(x)$ by first evaluating explicitly the case of $n = 0, 1$, and 2 [specify also the value s_0 such that the Laplace transform $F_n(s) = \mathscr{L}[x^n]$ is well-defined for all $s > s_0$ and all integer $n \geq 0$]. (11%)

(2) determine the Fourier transform for $f_0(x)$; does the Fourier transform of $f_n(x)$ exist for $n \geq 1$? (5%)

6. The Legendre polynomial reads as $P_\ell(x) = \frac{1}{2^\ell \ell!}v(x)$ where $v(x) = \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}[(x^2-1)^\ell]$

(1) Show that $v(x)$ and (hence) $P_\ell(x)$ are both solutions to $(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \ell(\ell+1)y = 0$. (8%)

(2) Show that $P_\ell(x)$ satisfies the orthogonality relation $\int_{-1}^{1} P_k(x)P_\ell(x)\mathrm{d}x = 0$ for all $k \neq \ell$. (7%)