國立成功大學

112學年度碩士班招生考試試題

編　號：37

系　所：物理學系

科　目：物理數學

日　期：0207

節　次：第 1 節

備　註：不可使用計算機

※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (a) Assume that a set of matrices satisfies the commutation relation
$$[M_j, M_k] = i\, M_\ell,$$
where $j, k,$ and ℓ are cyclic indices. Show that the trace of each matrix vanishes. (5 points)
(b) A and B are two non-commuting Hermitian matrices: $AB - BA = i\,C$ with $i = \sqrt{-1}$. Prove that C is Hermitian. (5 points)

2. Vector **B** is formed by the product of two gradients: $\boldsymbol{B} = (\nabla u) \times (\nabla \mathrm{v})$, where u and v are scalar functions.
(a) Show that $\boldsymbol{B}$ is solenoidal. (10 points)
(b) Show that $\boldsymbol{A} = \frac{1}{2}(u\,\nabla \mathrm{v} - \mathrm{v}\,\nabla u)$ is a vector potential for **B** defined above. (15 points)

3. Solve the ordinary differential equation
$$\frac{dy}{dx} + \left(1 + \frac{y}{x}\right) = 0. \quad \text{(15 points)}$$

4. (a) State the Cauchy's integral theorem. (5 points)
(b) State the Cauchy's integral formula. (5 points)
(c) Evaluate
$$\oint \frac{dz}{z(2z+1)}$$
for the contour around the unit circle. (10 points)

5. Gamma function is defined as: $\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt$ with Re(z)>0.
(a) Show that $\int_0^\infty e^{-x^4}\, dx = \Gamma\left(\frac{5}{4}\right)$. (10 points)
(b) Show that $\lim_{x\to 0} \frac{\Gamma(a\,x)}{\Gamma(x)} = \frac{1}{a}$. (10 points)

6. Show that the Schwarz inequality:
$$|\langle f|g\rangle|^2 \le \langle f|f\rangle\,\langle g|g\rangle,$$
where $|f\rangle$ and $|g\rangle$ are the vectors in the Hilbert space. (10 points)