(10%) 1. Define a linear vector space.

(10%) 2. Proof the operator $i\frac{\partial}{\partial x}$ is hermitian.

(20%) 3. find $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{x^2 - \sigma^2}$

(20%) 4. From $\frac{d^2y}{dx^2} + y = 0$, let one of the solutions be $y_1 = \sin x$.

(a) Find the other solution by using "method of variation".

(b) Show these two solutions are linear independent.

(20%) 5. The P_n are the Legendre polynomial defined by $g(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$, $|t|<1$

Show (a)

$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$, $n=1,2,\ldots$

(b) $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$

(c) $P_n(-x) = (-1)^n P_n(x)$

(20%) 6. Let $M = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$

$$M' = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix}$$

$$U = \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{-i\alpha}{2}} \end{pmatrix}$$

$$M' = U M U^{\dagger}$$

(a) find the transformation matrix between (x', y', z') and (x, y, z). i.e. $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(b) what is the meaning of U and R.

(c) what kind of mapping exsits between U and R.