系所組別： 物理學系
考試科目： 古典力學

※ 考生請注意：本試題 □可 ☑不可 使用計算機

1. As in fig. 1, a rope of uniform linear density ρ and length $2\pi R$ is wrapped around a cylinder of radius R and rotational inertia $I = \beta MR^2$. The cylinder rotates freely about its axis. The rope ends are initially at $\theta = 0$ (one fixed, one loose) and at rest. It is then slightly disturbed to rotate. Find the angular velocity $\omega(\theta)$. (20 points) (Hint: Wrap the rope back to the cylinder and consider the potential energy for a segment dx.)
2. A space craft (m) is designed to dispose of nuclear waste by crashing into the Sun (M) along an elliptical orbit with minimum distance R_s (Sun's radius) and maximum distance r_E (radius of Earth's orbit) from the Sun's center. (a) What initial velocity Δv of the craft relative to the Earth is needed? (b) How much time, in unit of year, will it take for the craft to reach the Sun? (10+5 points)
3. As in fig. 3, the point of support of a simple pendulum of length b is fixed on a rim of radius a rotating with constant angular velocity ω. Take θ as the coordinate, write down the Lagrangian L, Hamiltonian H and the equation of motion for θ. (20 points)
4. See fig. 4 for a Foucault pendulum of length l, at a latitude λ, under the gravitational force $m\vec{g}$, string tension $\vec{T}$ $(T \approx mg)$ and Coriolis force $m2\vec{v}\times\vec{\omega}$, where $\vec{\omega} = (\omega_x, \omega_y, \omega_z) = (-\omega\cos\lambda, 0, \omega\sin\lambda)$ is the earth's spin angular velocity. (a) Derive the eqs. of motion $\ddot{x} + \alpha^2 x \approx 2\omega_z \dot{y}$ & $\ddot{y} + \alpha^2 y \approx -2\omega_z \dot{x}$, where $\alpha \equiv g/l$. (b) For $\alpha >> \omega_z$, the eqs. are solved by $q(t) \equiv x(t) + iy(t) \approx e^{-i\omega_z t}(Ae^{i\alpha t} + Be^{-i\alpha t})$. Explain why this $q(t)$ implies that the pendulum's plane rotates with angular velocity ω_z. (10+10 points)
5. (a) An oscillating system has kinetic energy $T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k$ and potential energy $U = \frac{1}{2}\sum_{j,k} A_{jk} q_j q_k$. Write down the eq. of motion, and show that the r-th normal mode $q_{jr}(x,t) = \mathrm{Re}\left[a_{jr} e^{i(\omega_r t - \delta_r)}\right]$ must satisfiy the eigenvalue eq. $\sum_j (A_{jk} - \omega_r^2 m_{jk}) a_{jr} = 0$。(b) As in fig. 5, two pendula of equal lengths b and equal masses m are connected by a spring of constant k. The spring is unstretched when $\theta_1 = 0 = \theta_2$. Given that $T = \frac{1}{2}m(b\dot{\theta}_1)^2 + \frac{1}{2}m(b\dot{\theta}_2)^2$ & $U \approx \frac{1}{2}mgb(\theta_1^2 + \theta_2^2) + \frac{1}{2}kb^2(\theta_1 - \theta_2)^2$ for small oscillation, find the eigenfrequencies $\omega_1^2 = g/b$ & $\omega_2^2 = g/b + 2\kappa/m$, and the normal coordinates $\eta_1 = \frac{1}{2a_{11}}(\theta_1 + \theta_2)$ & $\eta_2 = \frac{1}{2a_{22}}(\theta_2 - \theta_1)$。 (10+15 points)

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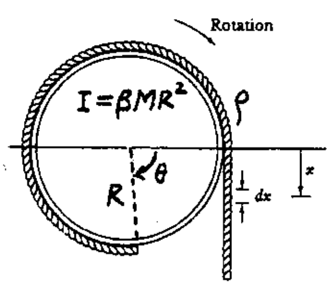

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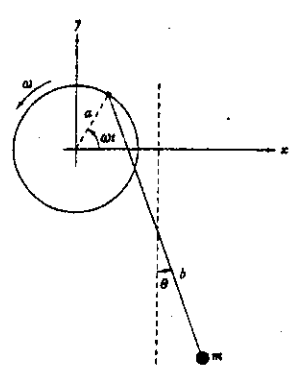

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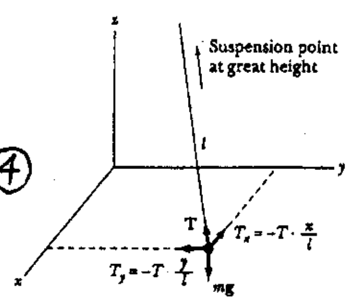

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