

系所組別： 數學系應用數學碩士班

考試科目： 高等微積分

考試日期： 0224，節次： 3

※ 考生請注意：本試題不可使用計算機

1. (10) Show that the sequence
- $\{z_n\}$
- defined by

$$z_n = \int_1^n \frac{\cos(x)}{x^2} dx$$

is Cauchy.

2. (10) Find
- $\lim_{n \rightarrow \infty} x_n$
- , where

$$x_n = \sum_{k=1}^n \frac{n^2}{\sqrt{n^6 + k}}, \quad n = 1, 2, \dots$$

(Hint : Squeeze theorem)

3. (20) Let
- $\{f_n\}_{n=1}^{\infty}$
- be a sequence of continuous functions. Prove or disprove the following:

- (a) (10) If $\{f_n\}$ converges to f , then f is continuous.
 (b) (10) If $\{f_n\}$ converges to f uniformly, then f is continuous.

4. (10)

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Show that f is continuous at $x = 0$.

5. (10) Show that the equation
- $e^x = 1 - x$
- has only one solution in
- \mathbb{R}
- . Find this solution.

6. (10) Show that

$$\int_0^{\pi} x e^{\sin(x)} dx = \frac{\pi}{2} \int_0^{\pi} e^{\sin(x)} dx$$

(Hint: substitute $u = \pi - x$)

7. (15) Does the integral

$$\int_1^{\infty} \frac{\sin(x)}{x} dx$$

converge absolutely ?

8. (15) Show that if a Cauchy sequence has a convergent subsequence, then it is convergent.