

※ 考生請注意：本試題不可使用計算機。請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (10%) Assume $f : [0, 1] \rightarrow [0, 1]$ is continuous. Prove that f has a fixed point in $[0, 1]$, i.e., there exists a point $x_1 \in [0, 1]$ such that $f(x_1) = x_1$.

2. (10%) Find

$$\lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{k}{n^2 + k^2} \right].$$

3. (12%) For each $n \in \mathbb{N}$ and each $x \in \mathbb{R}$, define

$$f_n(x) = \frac{1 - |x|^n}{1 + |x|^n}.$$

Prove that the sequence $\{f_n\}$ converges pointwise but not uniformly on \mathbb{R} .

4. (12%) Prove that $f(x) = \sin(x^2)$ is not uniformly continuous on $[0, \infty)$.

5. (12%) Let

$$f(x, y) = \begin{cases} \frac{x^2 y}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Is f is differentiable at $(0, 0)$?

6. (14%) For what value of $a > 1$ is

$$\int_a^{a^2} \frac{1}{x} \ln \left(\frac{x-1}{32} \right) dx$$

minimum?

7. (15%) Let $F : \mathbb{R}^5 \rightarrow \mathbb{R}^2$ be defined by $F(u, v, w, x, y) = (uy + vx + w + x^2, uvw + x + y + 1)$, and $F(2, 1, 0, -1, 0) = (0, 0)$.

(a) (7%) Show that we can solve $F(u, v, w, x, y) = (0, 0)$ for (x, y) in terms of (u, v, w) near $(2, 1, 0)$.

(b) (8%) If $(x, y) = \varphi(u, v, w)$ is the solution of the preceding part, compute $D\varphi(2, 1, 0)$.

8. (15%) Let

$$f(x, y) = \begin{cases} 0, & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 1, & \text{if either } x = 0 \text{ or } y = 0, \end{cases} \quad \text{defined on } \mathbb{R}^2.$$

Show that $\lim_{x \rightarrow a} f(x, y)$ and $\lim_{y \rightarrow b} f(x, y)$ both exist, and

$$\lim_{x \rightarrow a} \left[\lim_{y \rightarrow b} f(x, y) \right] = \lim_{y \rightarrow b} \left[\lim_{x \rightarrow a} f(x, y) \right] = L$$

for all $a, b \in \mathbb{R}$. Does $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ exist?