國立成功大學 105 學年度碩士班招生考試試題

系　所：數學系應用數學碩士班

考試科目：線性代數　　考試日期：0228，節次：1

※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

Unless otherwise specified, all the matrices are assume to be complex matrices. The field of real numbers is denoted by $\mathbb{R}$ and the field of complex numbers is denoted by $\mathbb{C}$. We also use I_n to denote the $n \times n$ identity matrix, while $\mathbf{0}$ denotes the zero vector.

1. Prove or give a counterexample for each of the following statements.
   (a) (5 points) Let A be a $n \times n$ matrix. If $A^k = I_n$ for some positive integer k, then A is invertible.
   (b) (5 points) Let A, B be two $m \times n$ matrices. If both systems $Ax = \mathbf{0}$ and $Bx = \mathbf{0}$ have nontrivial solutions, then $(A+B)x = \mathbf{0}$ has nontrivial solutions.
   (c) (5 points) If $v_1, \ldots, v_n$ are linearly independent, then $T(v_1), \ldots, T(v_n)$ are linearly independent, where $T : V \to W$ is a linear transformation and $v_i \in V$.
   (d) (5 points) If $T(v_1), \ldots, T(v_n)$ are linearly independent, then $v_1, \ldots, v_n$ are linearly independent, where $T : V \to W$ is a linear transformation and $v_i \in V$.

2. (20 points) Find all possible real number x_1, x_2, x_3, x_4 and x_5 that satisfy the following system of linear equations:

$$\begin{cases} x_1 & -x_2 & +x_3 & +2x_4 & -x_5 & = -1 \\ 2x_1 & +x_2 & +2x_3 & -x_4 & +x_5 & = 2 \\ 4x_1 & -x_2 & +4x_3 & +3x_4 & -x_5 & = 0. \end{cases}$$

3. (20 points) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix, where A is given by

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

4. Let $T : V \to V$ be a linear operator on a finite dimensional complex inner product space V. The adjoint of T is the linear operator $T^* : V \to V$ such that $< T(v), w > = < v, T^*(w) >$ for all $v, w \in V$.
   (a) (5 points) Suppose $\lambda \in \mathbb{C}$. Prove that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*. (We use $\overline{\lambda}$ to denote the complex conjugate of λ.)
   (b) (5 points) If T is self-adjoint, that is, $T = T^*$, prove that every eigenvalue of T is real.
   (c) (5 points) Show that every eigenvalue of T^*T is a positive real number.
   (d) (5 points) If T is normal, that is, $T^*T = TT^*$, prove that $\operatorname{Ker} T = \operatorname{Ker} T^*$. Recall that the kernel of T is defined as $\operatorname{Ker} T = \{v \in V \mid T(v) = \mathbf{0}\}$.
   (e) (5 points) Prove that if T is normal, then $\operatorname{Ker} T^k = \operatorname{Ker} T$ for all positive integer k.

5. (8 points) Let

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Show that every eigenvalue of A is a positive real number. (Hint: It suffices to show that A is positive definite.)

6. (7 points) Let A be an $n \times n$ Hermitian matrix (that is, $A = A^*$, where A^* denotes its conjugate transpose) satisfying the condition $A^5 + A^3 + A = 3I_n$. Show that $A = I_n$. (Hint: Use the fact that every eigenvalue of A is real, and consider the minimal polynomial.)